

n -groupoids and stacky groupoids

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June 29, 2009

Abstract

We discuss two generalizations of Lie groupoids. One consists of Lie n -groupoids defined as simplicial manifolds with trivial $\pi_{k \geq n+1}$. The other consists of stacky Lie groupoids $\mathcal{G} \Rightarrow M$ with \mathcal{G} a differentiable stack. We build a 1–1 correspondence between Lie 2-groupoids and stacky Lie groupoids up to a certain Morita equivalence. We prove this in a general set-up so that the statement is valid in both differential and topological categories. Hypercovers of higher groupoids in various categories are also described.

KEY WORDS stacks, groupoids, simplicial objects, Morita equivalence

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*Research partially supported by the Liftoff fellowship 2004 of the Clay Institute

1 Introduction

Recently there has been much interest in higher group(oid)s, which generalize the notion of group(oid)s in various ways. Some of them turn out to be unavoidable to study problems in differential geometry. An example comes from the string group, which is a 3-connected cover of $Spin(n)$. More generally, to any compact simply connected group G one can associate its string group $String_G$. It has various models, given by Stolz and Teichner [33, 34] using an infinite-dimensional extension of G , by Brylinski [9] using a $U(1)$ -gerbe with the connection over G , and recently by Baez et al. [4] using Lie 2-groups and Lie 2-algebras. Henriques [21] constructs the string group as a higher group that we study in this paper and as an integration object of a certain Lie 2-algebra with an integration procedure which is also studied in [18, 39, 42].

Other examples come from a kind of étale stacky groupoid (called a Weinstein groupoid) [36] built upon the very important work of [12, 13]. These stacky groupoids are the global objects in 1–1 correspondence with Lie algebroids. A Lie algebroid can be understood as a degree-1 super manifold with a degree-1 homological vector field, or more precisely as a vector bundle $A \rightarrow M$ equipped with a Lie bracket $[\cdot, \cdot]$ on the sections of A and a vector bundle morphism $\rho : A \rightarrow TM$, satisfying a Leibniz rule,

$$[X, fY] = f[X, Y] + \rho(X)(f)Y.$$

When the base M is a point, the Lie algebroid becomes a Lie algebra. Notice that unlike (finite-dimensional) Lie algebras which always have associated Lie groups, Lie algebroids do not always have associated Lie groupoids [2, 3]. One needs to enter the world of stacky groupoids to obtain the desired 1–1 correspondence. Since Lie algebroids are closely related to Poisson geometry, this result applies to complete the first step of Weinstein’s program of quantization of Poisson manifolds: to associate to Poisson manifolds their symplectic groupoids [40, 41]. It turns out that some “non-integrable” Poisson manifolds cannot have symplectic (Lie) groupoids. A solution to this problem is given in [37] with the above result so that every Poisson manifold has a corresponding stacky symplectic groupoid.

2-group(oid)s were already studied in the early twentieth century by Whitehead and his followers under various terms, such as crossed modules. They are also studied from the aspect of “gr-champ” (i.e. stacky groups) by Breen [7]. Recently, various versions of 2-groups, with different strictness, have been studied by Baez’s school (the best thing is to read their n-category café on <http://golem.ph.utexas.edu/category/>). These authors also study a lot of developments on the subjects surrounding 2-groups such as 2-bundles, 2-connections and the relation with gerbes.

It seems that it is required now to have a uniform method to describe 2-groups so that it opens a way to treat all higher groupoids. In this paper, we apply a simplicial method to describe all higher groupoid objects in various categories in an elegant way, and prove when $n = 2$, they are the same as stacky groupoid objects in these category. This idea (set theoretically) was known much earlier by Duskin and Glenn [14, 19]. The 0-simplices correspond to the objects, the 1-simplices correspond to the arrows (or 1-morphisms), and the higher dimensional simplices correspond to the higher morphisms. This method becomes much more suitable when dealing with the differential or topological category.

Recall that a simplicial set (respectively manifold) X is made up of sets (respectively

manifolds) X_n and structure maps

$d_i^n : X_n \rightarrow X_{n-1}$ (face maps) $s_i^n : X_n \rightarrow X_{n+1}$ (degeneracy maps), for $i \in \{0, 1, 2, \dots, n\}$

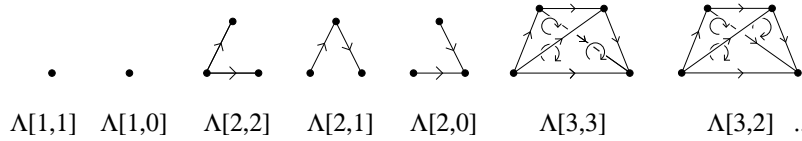
that satisfy the coherence conditions

$$\begin{aligned} d_i^{n-1} d_j^n &= d_{j-1}^{n-1} d_i^n \text{ if } i < j, & s_i^n s_j^{n-1} &= s_{j+1}^n s_i^{n-1} \text{ if } i \leq j, \\ d_i^n s_j^{n-1} &= s_{j-1}^{n-2} d_i^{n-1} \text{ if } i < j, & d_j^n s_j^{n-1} &= id = d_{j+1}^n s_j^{n-1}, & d_i^n s_j^{n-1} &= s_j^{n-2} d_{i-1}^{n-1} \text{ if } i > j + 1. \end{aligned} \quad (1)$$

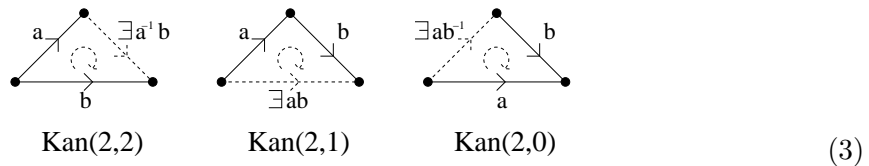
The first two examples of simplicial sets are the simplicial m -simplex $\Delta[m]$ and the horn $\Lambda[m, j]$ with

$$\begin{aligned} (\Delta[m])_n &= \{f : (0, 1, \dots, n) \rightarrow (0, 1, \dots, m) \mid f(i) \leq f(j), \forall i \leq j\}, \\ (\Lambda[m, j])_n &= \{f \in (\Delta[m])_n \mid \{0, \dots, j-1, j+1, \dots, m\} \not\subseteq \{f(0), \dots, f(n)\}\}. \end{aligned} \quad (2)$$

In fact the horn $\Lambda[m, j]$ is a simplicial set obtained from the simplicial m -simplex $\Delta[m]$ by taking away its unique non-degenerate m -simplex as well as the j -th of its $m+1$ non-degenerate $(m-1)$ -simplices, as in the following picture (in this paper all the arrows are oriented from bigger numbers to smaller numbers):



A simplicial set X is *Kan* if any map from the horn $\Lambda[m, j]$ to X ($m \geq 1, j = 0, \dots, m$), extends to a map from $\Delta[m]$. Let us call $Kan(m, j)$ the Kan condition for the horn $\Lambda[m, j]$. A *Kan simplicial set* is therefore a simplicial set satisfying $Kan(m, j)$ for all $m \geq 1$ and $0 \leq j \leq m$. In the language of groupoids, the Kan condition corresponds to the possibility of composing and inverting various morphisms. For example, the existence of a composition for arrows is given by the condition $Kan(2, 1)$, whereas the composition of an arrow with the inverse of another is given by $Kan(2, 0)$ and $Kan(2, 2)$.



Note that the composition of two arrows is in general not unique, but any two of them can be joined by a 2-morphism h given by $Kan(3, 2)$.

Here, h ought to be a bigon, but since we do not have any bigons in a simplicial set, we view it as a triangle with one of its edges degenerate. The degenerate 1-simplex above z is denoted 1_z .

In an n -groupoid, the only well-defined composition law is the one for n -morphisms. This motivates the following definition.

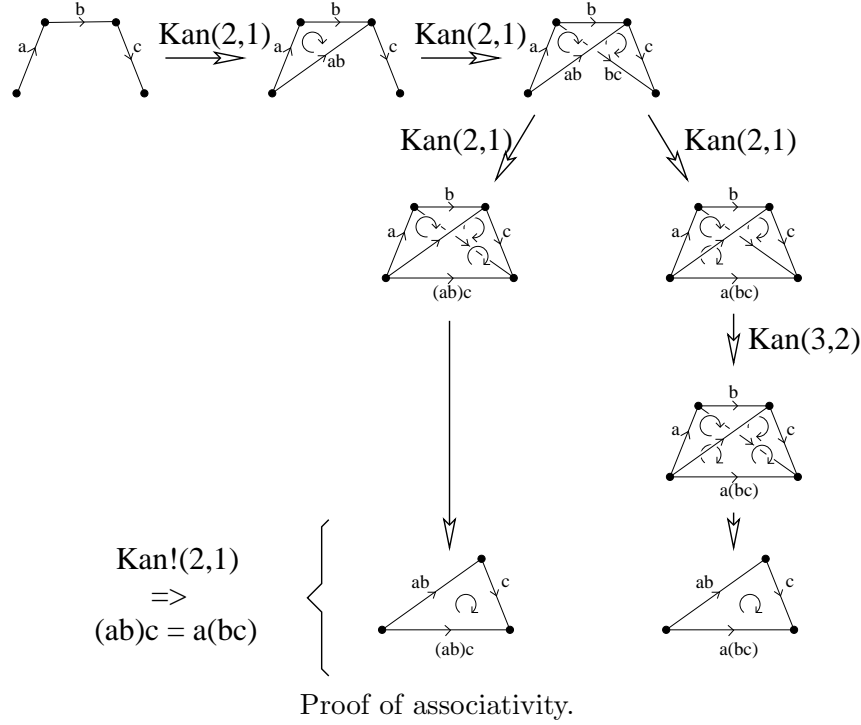
Definition 1.1. An n -groupoid ($n \in \mathbb{N} \cup \infty$) X is a simplicial set that satisfies $Kan(m, j)$ for all $0 \leq j \leq m \geq 1$ and $Kan!(m, j)$ for all $0 \leq j \leq m > n$, where

$Kan(m, j)$: Any map $\Lambda[m, j] \rightarrow X$ extends to a map $\Delta[m] \rightarrow X$.

$Kan!(m, j)$: Any map $\Lambda[m, j] \rightarrow X$ extends to a unique map $\Delta[m] \rightarrow X$.

An n -group is an n -groupoid for which X_0 is a point. When $n = 2$, they are different from the various kinds of 2-group(oid)s or double groupoids in [5, 8] (see [20, Appendix] for an explanation of the relation between our 2-group and the one in [5]), and are not exactly the same as in [30], as he requires a choice of composition and strict units; however, they are the same as in [14]. A usual groupoid (category with only isomorphisms) is equivalent to a 1-groupoid in the sense of Def. 1.1. Indeed, from a usual groupoid, one can form a simplicial set whose n -simplices are given by sequences of n composable arrows. This is a standard construction called the *nerve* of a groupoid and one can check that it satisfies the required Kan conditions.

On the other hand, a 1-groupoid X in the sense of Def. 1.1 gives us a usual groupoid with objects and arrows given respectively by the 0-simplices and 1-simplices of X . The unit is provided by the degeneracy $X_0 \rightarrow X_1$, the inverse and composition are given by the Kan conditions $Kan(2, 0)$, $Kan(2, 1)$ and $Kan(2, 2)$ as in (3), and the associativity is given by $Kan(3, 2)$ and $Kan!(2, 1)$.



This motivates the corresponding definition in a category \mathcal{C} with a singleton Grothendieck pretopology \mathcal{T} which satisfies some additional mild assumptions (see Assumptions 2.1). We shall assume \mathcal{C} has all coproducts.

Definition 1.2. A *singleton Grothendieck pretopology*¹ on \mathcal{C} is a collection of morphisms, called covers, subject the following three axioms: Isomorphisms are covers. The composition of two covers is a cover. If $U \rightarrow X$ is a cover and $Y \rightarrow X$ is a morphism, then the pull-back $Y \times_X U$ exists, and the natural morphism $Y \times_X U \rightarrow Y$ is a cover.

We list examples of categories equipped with singleton Grothendieck pretopologies in Table 1, among which $(\mathcal{C}_i, \mathcal{T}'_i)$ for $i = 1, 2, 3$ satisfy Assumptions 2.1 (with the terminal object $*$ being a point).

Definition 1.3. [21] An n -groupoid object X ($n \in \mathbb{N} \cup \infty$) in $(\mathcal{C}, \mathcal{T})$ is a simplicial object in $(\mathcal{C}, \mathcal{T})$ that satisfies $\text{Kan}(m, j)$ for all $0 \leq j \leq m \geq 1$ and $\text{Kan}!(m, j)$ $0 \leq j \leq m > n$, where

$\text{Kan}(m, j)$: The restriction map $\text{hom}(\Delta[m], X) \rightarrow \text{hom}(\Lambda[m, j], X)$ is a cover in $(\mathcal{C}, \mathcal{T})$.

$\text{Kan}!(m, j)$: The restriction map $\text{hom}(\Delta[m], X) \rightarrow \text{hom}(\Lambda[m, j], X)$ is an isomorphism in \mathcal{C} .

The notation $\text{hom}(S, X)$, when S is a simplicial set and X is a simplicial object in \mathcal{C} , has the same meaning as in [21, Section 2]; in the case of a **Lie n -groupoid** [21, Def. 1.2], which

¹The original definition of Grothendieck pretopology [1] requires a collection of morphisms $U_i \rightarrow X$ for a cover. But since we assume that \mathcal{C} has coproducts, a Grothendieck pretopology is given by a singleton Grothendieck pretopology by declaring $\{U_i \rightarrow X\}$ to be a cover if $\Pi_i U_i \rightarrow X$ is a cover. Hence when coproducts exist, these two concepts are the same.

²See [21, Section 4] for the convention on Banach manifolds that we use.

is an n -groupoid object in $(\mathcal{C}, \mathcal{T}) = (\mathcal{C}_1, \mathcal{T}'_1)$, we can view simplicial sets $\Delta[m]$ and $\Lambda[m, j]$ as simplicial manifolds with their discrete topology so that $\text{hom}(S, X)$ denotes the set of homomorphisms of simplicial manifolds with its natural topology. Thus $\text{hom}(\Delta[m], X)$ is just another name for X_m . However it is not obvious that $\text{hom}(\Lambda[m, j], X)$ is still an object in \mathcal{C} , and it is a result of [21, Corollary 2.5] (see Section 2 for details). Moreover, a Lie n -group is a Lie n -groupoid X where $X_0 = pt$.

On the other hand, a *stacky Lie (SLie) groupoid* $\mathcal{G} \Rightarrow M$, following the concept of Weinstein (W-) groupoid in [36], is a groupoid object in the world of differentiable stacks with its base M an honest manifold. When \mathcal{G} is also a manifold, $\mathcal{G} \Rightarrow M$ is obviously a Lie groupoid. *W-groupoids*, which are *étale SLie* groupoids, provide a way to build the 1–1 correspondence with Lie algebroids. This concept can be also adapted to stacky groupoids in various categories (see Def. 3.4).

Given these two higher generalizations of Lie groupoids, Lie n -groupoids and SLie groupoids, arising from different motivations and constructions, we ask the following questions:

- Are SLie groupoids the same as Lie n -groupoids for some n ?
- If not exactly, to which extent they are the same?
- Is there a way to also realize Lie n -groupoids as integration objects of Lie algebroids?

In this paper, we answer the two first questions by

Theorem 1.4. *There is a one-to-one correspondence between SLie (respectively W-) groupoids and Lie 2-groupoids (respectively Lie 2-groupoids whose X_2 is étale over $\text{hom}(\Lambda[2, j], X)$) modulo 1-Morita equivalences³ of Lie 2-groupoids.*

The last question will be answered positively in a future work [42]:

Theorem 1.5. *Let A be a Lie algebroid and let $L\text{mor}(-, -)$ be the space of Lie algebroid homomorphisms satisfying suitable boundary conditions. Then*

$$L\text{mor}(T\Delta^2, A)/L\text{mor}(T\Delta^3, A) \Rightarrow L\text{mor}(T\Delta^1, A) \Rightarrow L\text{mor}(T\Delta^0, A),$$

is a Lie 2-groupoid corresponding to the W-groupoid $\mathcal{G}(A)$ constructed in [36] under the correspondence in the above theorem.

³Morita equivalences preserving X_0

Table 1: Categories and pretopologies

Notation	\mathcal{C}	cover
$(\mathcal{C}_1, \mathcal{T}_1)$	Banach manifolds ² and smooth morphisms	surjective étale morphisms
$(\mathcal{C}_1, \mathcal{T}'_1)$	Banach manifolds and smooth morphisms	surjective submersions
$(\mathcal{C}_2, \mathcal{T}_2)$	Topological spaces and continuous morphisms	surjective étale morphisms
$(\mathcal{C}_2, \mathcal{T}'_2)$	Topological spaces and continuous morphisms	surjective continuous morphisms
$(\mathcal{C}_3, \mathcal{T}_3)$	Affine schemes and smooth morphisms	surjective étale morphisms
$(\mathcal{C}_3, \mathcal{T}'_3)$	Affine schemes and smooth morphisms	surjective smooth morphisms

With a mild assumption about “good charts”, we are able to prove a stronger version of Theorem 1.4 in various other categories, such as topological categories (see Theorem 4.8). If we view a manifold as a set with additional structure, then we can view our SLie groupoid $\mathcal{G} \Rightarrow M$ as a groupoid where the space \mathcal{G} of arrows is itself a category with certain additional structure. From this viewpoint, our result is the analogue in geometry of Duskin’s result [15] in category theory. Moreover, our stacks are required to be presentable by certain charts in \mathcal{C} . For example, when $(\mathcal{C}, \mathcal{T}) = (\mathcal{C}_1, \mathcal{T}'_1)$ the differential category, our stacks are *not* just categories fibred in groupoids over \mathcal{C}_1 , but furthermore can be presented by Lie groupoids. They are called differentiable stacks. Hence to prove our result, we use the equivalence of the 2-category of differentiable stacks, morphisms and 2-morphisms and the 2-category of Lie groupoids, Hilsum–Skandalis (H.S.) bibundles [28, 26] and 2-morphisms. This can be viewed as an enrichment of Duskin’s set-theoretical method. Then of course, this enrichment requires a different approach and solutions of many technical issues in geometry and topology that we prepare in Section 2 and 3.

Furthermore, a subtle point in the theory of stacks and groupoids is that a stack can be presented by many Morita equivalent groupoids. Hence, for Theorem 1.4 and 4.8, we also develop the theory of *morphisms* and *Morita equivalence* of n -groupoids, which is expected to be useful in the theory of n -stacks and n -gerbes and should correspond to Morita equivalence of stacky groupoids in [10] when $n = 2$.

The reader’s first guess about the morphisms of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$ is probably that a morphism $f : X \rightarrow Y$ ought to be a simplicial morphism, namely a collection of morphisms $f_n : X_n \rightarrow Y_n$ in \mathcal{C} that commute with faces and degeneracies. In the language of categories, this is just a natural transformation from the functor X to the functor Y . We shall call such a natural transformation a *strict map* from X to Y . Unfortunately, it is known that, already in the case of usual Lie groupoids, such strict notions are not good enough. Indeed there are strict maps that are not invertible even though they ought to be isomorphisms. That’s why people introduced the notion of H.S. bibundles. Here is an example of such a map: consider a manifold M with an open cover $\{\mathcal{U}_\alpha\}$. The simplicial manifold X with $X_n = \bigsqcup_{\alpha_1, \dots, \alpha_n} \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_n}$ maps naturally to the constant simplicial manifold M . All the fibers of that map are simplices, in particular they are contractible simplicial sets. Nevertheless, that map has no inverse.

The second guess is then to define a special class of strict maps which we shall call *hypercovers*. A map from X to Y would then be a *zig-zag* of strict maps $X \xleftarrow{\sim} Z \rightarrow Y$, where the map $Z \xrightarrow{\sim} X$ is one of these hypercovers. This will be equivalent to bibundle approach. The notion of hypercover is nevertheless very useful (e.g., to define sheaf cohomology of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$) and we study it in Section 2.1.

We also find some technical improvements of the concept of SLie groupoid: it turns out that an SLie groupoid $\mathcal{G} \Rightarrow M$ always has a “good groupoid presentation” G of \mathcal{G} , which possesses a strict groupoid map $M \rightarrow G$. Moreover the condition on the inverse map can be simplified.

Notation Chart

$\mathcal{G} \Rightarrow M, \mathcal{H} \Rightarrow N$: stacky groupoids;

$\bar{s}, \bar{t}, \bar{e}, \bar{i}, m$: source, target, identity, inverse and multiplication of a stacky groupoid;

$G := G_1 \Rightarrow G_0$: a groupoid presentation of \mathcal{G} ;

s_G, t_G, e_G, i_G : the source, target, identity and inverse of the groupoid G respectively;

$\mathbf{s}, \mathbf{t} : G_0 \rightarrow M$: the morphisms presenting $\bar{\mathbf{s}}, \bar{\mathbf{t}} : \mathcal{G} \rightarrow M$ respectively;
 η_i, γ_i : the face facing the vertex i , moreover γ_i belongs to G_1 ;
 η_{ijk}, γ_{ijk} : the face with vertices i, j and k , moreover γ_{ijk} belongs to G_1 ;
 J_l, J_r : the left and the right moment maps⁴ of an H.S. bibundle E between two groupoid objects $K_1 \rightrightarrows K_0$ and $K'_1 \rightrightarrows K'_0$,

$$\begin{array}{ccccc}
 & K_1 & & E & & K'_1 \\
 & \Downarrow & & \swarrow J_l & & \searrow J_r \\
 & K_0 & & & & K'_0
 \end{array}$$

Acknowledgments: Here I would like to thank Henrique Bursztyn and Alan Weinstein for their hosting and very helpful discussions. I also thank Laurent Bartholdi, Marco Zambon and Toby Bartels for many editing suggestions. I especially thank André Henriques who pointed out to me the potential correspondence of stacky groupoids and Lie 2-groupoids during the conference of “Groupoids and Stacks in Physics and Geometry” in CIRM-Luminy 2004. I owe a lot to discussions with André. He contributed the exact definitions of Lie n -groupoids and their hypercovers, and also nice pictures! Also I thank Ezra Getzler very much for telling me the relation of his work [18], n -groupoids and our work [36] during my trip to Northwestern and his continuous comments to this work later on. I thank John Baez and Toby Bartels for discussions on Grothendieck pretopologies. Finally, I thank the referee a lot for much helpful advice.

2 n -groupoid objects and morphisms in various categories

Lie groupoids and topological groupoids have been studied a lot (see [11] for details). They are used to study foliations, and more recently orbifolds, differentiable stacks and topological stacks [27, 6, 30, 17]. Here we will try to convince the reader that it is fruitful to consider them within the context of n -groupoid objects (Def. 1.3), especially if one wants to define and use sheaf cohomology.

Our n -groupoid objects live in a category \mathcal{C} with a singleton Grothendieck pretopology \mathcal{T} satisfying the following properties:

Assumptions 2.1. *The category \mathcal{C} has a terminal object $*$, and for any object $X \in \mathcal{C}$, the map $X \rightarrow *$ is a cover.*

The pretopology \mathcal{T} is subcanonical, which means that all the representable functors $T \mapsto \text{hom}(T, X)$ are sheaves.

Remark 2.2. These properties are (*only*) a part of Assumptions 2.2 in [21]. It turns out that we do not need all the assumptions if we do not deal with further subjects, such as simplicial homotopy groups.

⁴They are called moment maps for the following reason: when we have a Hamiltonian action of a Lie group K on a symplectic manifold E with a moment map $J : E \rightarrow \mathfrak{k}^*$, then the Lie groupoid $T^*K \rightrightarrows \mathfrak{k}$ acts on E with the help of the map $E \xrightarrow{J} \mathfrak{k}^*$; then this result was generalized to any (symplectic) groupoid action in [25] keeping the name “moment map”.

As in [21, Section 2], we sometimes talk about the limit of a diagram in \mathcal{C} , before knowing its existence. For this purpose, we use the Yoneda functor

$$\begin{aligned} \text{yon} : \mathcal{C} &\rightarrow \{\text{Sheaves on } \mathcal{C}\} \\ X &\mapsto (T \mapsto \text{hom}(T, X)) \end{aligned}$$

to embed \mathcal{C} to the category of sheaves on \mathcal{C} . Hence a limit of objects of \mathcal{C} can always be viewed as the limit of the corresponding representable sheaves using yon . The limit sheaf is representable if and only if the original diagram has a limit in \mathcal{C} .

2.1 Hypercovers of n -groupoid objects

First let us fix some notation of pull-back spaces of the form $PB(\text{hom}(A, Z) \rightarrow \text{hom}(A, X) \leftarrow \text{hom}(B, X))$, where the maps are induced by some fixed maps $A \rightarrow B$ and $Z \rightarrow X$. To avoid the cumbersome pull-back notation, we shall denote these spaces by

$$\left\{ \begin{array}{ccc} A & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ B & \xrightarrow{?} & X \end{array} \right\} \text{ in the layout, or } \text{hom}(A \rightarrow B, Z \rightarrow X) \text{ in the text.}$$

This notation indicates that the space parameterizes all commuting diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & & \downarrow \\ B & \longrightarrow & X, \end{array}$$

where we allow the horizontal arrows to vary but we fix the vertical ones.

Hypercovers of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$ are very much inspired by hypercovers of étale simplicial objects [1, 16] and by Quillen's trivial fibrations for simplicial sets⁵ [32].

Definition 2.3. A strict map $f : Z \rightarrow X$ of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$ is a *hypercove* if the natural map from $Z_k = \text{hom}(\Delta^k, Z)$ to the pull-back

$$\text{hom}(\partial\Delta[k] \rightarrow \Delta[k], Z \rightarrow X) = PB(\text{hom}(\partial\Delta[k], Z) \rightarrow \text{hom}(\partial\Delta[k], X) \leftarrow X_k)$$

is a cover for $0 \leq k \leq n-1$ and an isomorphism⁶ for $k = n$.

But in our case, we need Lemma 2.4 to *justify* that $\text{hom}(\partial\Delta[k] \rightarrow \Delta[k], Z \rightarrow X)$ is an object in \mathcal{C} for $1 \leq k$ so that this definition makes sense. This is specially surprising since the spaces $\text{hom}(\partial\Delta[m], Z)$ need not be in \mathcal{C} (for example take $n = 2$, \mathcal{C} the category of Banach manifolds, and Z the cross product Lie groupoid associated to the action of S^1 on \mathbb{R}^2 by rotation around the origin). To simplify our notation, \rightrightarrows and \leftrightsquigarrow always denote covers in \mathcal{T} .

⁵In fact, ∞ -groupoid objects in $(\mathcal{C}, \mathcal{T})$ are called *Kan simplicial objects* in $(\mathcal{C}, \mathcal{T})$ [21, Section 2].

⁶When $n = \infty$, namely in the case of ∞ -groupoid objects in $(\mathcal{C}, \mathcal{T})$, the requirement of isomorphism is empty.

Lemma 2.4. *Let S be a finite collapsible simplicial set⁷ of any dimension, and $T (\hookrightarrow S)$ a sub-simplicial set of dimension $\leq m$. Let $f : Z \rightarrow X$ be a strict map of ∞ -groupoid objects in $(\mathcal{C}, \mathcal{T})$ such that $\text{hom}(\partial\Delta[l] \rightarrow \Delta[l], Z \rightarrow X) \in \mathcal{C}$ for all $l \leq m$ and the natural map⁸*

$$Z_l \rightarrow \left\{ \begin{array}{ccc} \partial\Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[l] & \xrightarrow{?} & X \end{array} \right\}$$

is a cover for all $l \leq m$. Then the pull-back $\text{hom}(T \rightarrow S, Z \rightarrow X)$ exists in \mathcal{C} . Hence in particular, $\text{hom}(\partial\Delta[m+1] \rightarrow \Delta[m+1], Z \rightarrow X)$ exists in \mathcal{C} .

Proof. Let T' be a sub-simplicial set obtained by deleting one l -simplex from T (without its boundary, and $T' \rightarrow T$ includes the case of $\rightarrow \Delta[0]$). We have a push-out diagram

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & T \\ \uparrow & & \downarrow \\ \partial\Delta[l] & \hookrightarrow & \Delta[l]. \end{array}$$

Applying the functor $\text{hom}(- \rightarrow S, Z \rightarrow X)$, this gives a pull-back diagram

$$\begin{array}{ccc} \left\{ \begin{array}{ccc} T' & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} T & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} \\ \downarrow & & \downarrow \\ \left\{ \begin{array}{ccc} \partial\Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} \Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\}, \end{array}$$

which may be combined with the pull-back diagram

⁷See [21, Section 2].

⁸Since $Z_l = \text{hom}(\Delta[l], Z)$ maps naturally to $\text{hom}(\Delta[l], X)$ and $\text{hom}(\partial\Delta[l], Z)$, there is a natural map from Z_l to their fibre product $\text{hom}(\partial\Delta[l] \rightarrow \Delta[l], Z \rightarrow X)$.

$$\begin{array}{ccc}
\left\{ \begin{array}{ccc} \partial\Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} \Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} \\
\downarrow & \lrcorner & \downarrow \\
\left\{ \begin{array}{ccc} \partial\Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[l] & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} \Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[l] & \xrightarrow{?} & X \end{array} \right\} = Z_l
\end{array}$$

to give yet another pull-back diagram

$$\begin{array}{ccc}
\left\{ \begin{array}{ccc} T' & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} T & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{?} & X \end{array} \right\} \\
\downarrow & \lrcorner & \downarrow \\
\left\{ \begin{array}{ccc} \partial\Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[l] & \xrightarrow{?} & X \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{ccc} \Delta[l] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[l] & \xrightarrow{?} & X \end{array} \right\} = Z_l.
\end{array} \tag{5}$$

By induction on the size of T ([21, Lemma 2.4] implies the case when $T = \emptyset$) and the induction hypothesis, we may assume that the upper left and lower left spaces in (5) are known to be in \mathcal{C} . The bottom arrow is a cover by hypothesis. Therefore by the property of covers, the upper right space is also in \mathcal{C} , which is what we wanted to prove. \square

As a byproduct of Lemma 2.4, we have:

Lemma 2.5. *If $Z \rightarrow X$ is a hypercover of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$, then for a sequence of subsimplicial sets $T' \subset T \subset S$ where S is collapsible, the natural map $\text{hom}(T \rightarrow S, Z \rightarrow X) \rightarrow \text{hom}(T' \rightarrow S, Z \rightarrow X)$ is a cover in \mathcal{C} . In particular,*

1. *the natural map $\text{hom}(\partial\Delta[m] \rightarrow \Delta[m], Z \rightarrow X) \rightarrow X_m$ is a cover, when we choose $T' = \emptyset$, $T = \partial\Delta[m]$ and $S = \Delta[m]$;*
2. *the natural map $Z_m \rightarrow \text{hom}(\Lambda[m, j] \rightarrow \Delta[m], Z \rightarrow X)$ is a cover in \mathcal{C} , when we choose $(T \rightarrow S) = (\Delta[m] \xrightarrow{id} \Delta[m])$ and $(T' \rightarrow S) = (\Lambda[m, j] \hookrightarrow \Delta[m])$;*
3. *the natural map $\text{hom}(\Lambda[m, j], Z) \rightarrow \text{hom}(\Lambda[m, j], X)$ is a cover in \mathcal{C} , when we choose $(T \rightarrow S) = (\Lambda[m, j] \xrightarrow{id} \Lambda[m, j])$ and $T' = \emptyset$;*

4. *we have*

$$Z_k \cong \text{hom}(\partial\Delta[k] \rightarrow \Delta[k], Z \rightarrow X), \quad \forall k \geq n. \tag{6}$$

Proof. We use the same induction as in Lemma 2.4 and only have to notice that the lower lever map in (5) is a cover, hence so is the upper lever map. Since composition of covers is still a cover, we obtain the result by introducing a sequence of subsimplicial sets $T' = T_0 \subset T_1 \subset \cdots \subset T_{j-1} \subset T_j$, where each T_i is obtained from T_{i-1} by removing a simplex.

For item 4, we take $(T \rightarrow S) = (\partial\Delta[n+1] \hookrightarrow \Delta[n+1])$ and $(T' \rightarrow S) = (\Lambda[n+1, j] \hookrightarrow \Delta[n+1])$, and use the fact that the lower lever map in (5) is an isomorphism when $l = n$. We obtain

$$\begin{aligned} \text{hom}(\partial\Delta[n+1] \rightarrow \Delta[n+1], Z \rightarrow X) &\cong \text{hom}(\Lambda[n+1, j] \rightarrow \Delta[n+1], Z \rightarrow X) \\ &= \text{hom}(\Lambda[n+1, j], Z) \times_{\text{hom}(\Lambda[n+1, j], X)} X_{n+1} \cong Z_{n+1}, \end{aligned}$$

since $X_{n+1} \cong \text{hom}(\Lambda[n+1, j], X)$. Then inductively, we obtain the result for all $k \geq n$. \square

Lemma 2.6. *The composition of hypercovers is still a hypercover.*

Proof. This is easy to verify, and we leave it to the reader. \square

Lemma 2.7. *Given a strict map $f : Z \rightarrow X$ and a hypercover $f' : Z' \rightarrow X$, the fibre product $Z \times_X Z'$ of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$ is still an n -groupoid object in $(\mathcal{C}, \mathcal{T})$.*

Proof. We first notice that $Z \times_X Z'$ is a simplicial object (of sheaves on \mathcal{C}) with each layer $\text{hom}(\Delta[m], Z \times_X Z') = Z_m \times_{X_m} Z'_m$. We use an induction to show that $Z \times_X Z'$ is an n -groupoid object in \mathcal{C} . First when $n = 0$, $Z'_0 \twoheadrightarrow X_0$, hence $Z_0 \times_{X_0} Z'_0$ is representable in \mathcal{C} and $Z_0 \times_{X_0} Z'_0 \twoheadrightarrow *$.

Now assume that $\text{hom}(\Delta[k], Z \times_X Z') \twoheadrightarrow \text{hom}(\Lambda[k, j], Z \times_X Z')$ is a cover in \mathcal{C} for $0 \leq j \leq k < m$. By item 3 of Lemma 2.5, $\text{hom}(\Lambda[m, j], Z \times_X Z')$ is representable. When $m < n$, we need to show that $\text{hom}(\Delta[m], Z \times_X Z') \twoheadrightarrow \text{hom}(\Lambda[m, j], Z \times_X Z')$ is a cover in \mathcal{C} ; when $m \geq n$, we need to show that $\text{hom}(\Delta[m], Z \times_X Z') \cong \text{hom}(\Lambda[m, j], Z \times_X Z')$ is an isomorphism in \mathcal{C} . When $m < n$, applying $X_m \twoheadrightarrow \text{hom}(\Lambda[m, j], X)$ to the south-east corner of the following pull-back diagram in \mathcal{C} ,

$$\begin{array}{ccc} \text{hom}(\Lambda[m, j], Z \times_X Z') & \xrightarrow{\quad} & \text{hom}(\Lambda[m, j], Z') \\ \downarrow & \lrcorner & \downarrow \\ \text{hom}(\Lambda[m, j], Z) & \xrightarrow{\quad} & \text{hom}(\Lambda(m, j), X). \end{array}$$

By item 2 in Lemma 2.5 and the fact that Z is an n -groupoid object in $(\mathcal{C}, \mathcal{T})$, we have

$$\begin{aligned} Z'_m \twoheadrightarrow \text{hom}(\Lambda[m, j] \rightarrow \Delta[m], Z' \rightarrow X) &= \text{hom}(\Lambda[m, j], Z') \times_{\text{hom}(\Lambda(m, j), X)} X_m, \\ Z_m \twoheadrightarrow \text{hom}(\Lambda[m, j], Z). \end{aligned}$$

Thus by Lemma 2.9, we have that $\text{hom}(\Delta[m], Z \times_X Z') \twoheadrightarrow \text{hom}(\Lambda[m, j], Z \times_X Z')$ is a cover in \mathcal{C} , which completes the induction. When $m \geq n$, the three \twoheadrightarrow 's above becomes three \cong 's (see (6)). Hence the same proof concludes $\text{hom}(\Delta[m], Z \times_X Z') \cong \text{hom}(\Lambda[m, j], Z \times_X Z')$. \square

Lemma 2.8. *Given Z, Z' and X n -groupoid objects in (\mathcal{C}, T) , if $f : Z \rightarrow X$ is a hypercover and $Z'' = Z \times_X Z'$ is still an n -groupoid object in (\mathcal{C}, T) , the natural map $Z'' \rightarrow Z'$ is a hypercover.*

Proof. Apply $\text{hom}(\partial\Delta[m] \rightarrow \Delta[m], -)$ to the pull-back diagram

$$\begin{array}{ccc} \left\{ \begin{array}{c} Z' \times_X Z \\ \downarrow \text{pr}_2 \\ Z \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} Z' \\ \downarrow f' \\ X \end{array} \right\} \\ \downarrow & \lrcorner & \downarrow \\ \left\{ \begin{array}{c} Z \\ \downarrow \text{id} \\ Z \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} X \\ \downarrow \text{id} \\ X \end{array} \right\}. \end{array}$$

We obtain a pull-back diagram in \mathcal{C} ,

$$\begin{array}{ccc} \left\{ \begin{array}{ccc} \partial\Delta[m] & \xrightarrow{?} & Z' \times_X Z \\ \downarrow & & \downarrow \\ \Delta[m] & \xrightarrow{?} & Z \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{ccc} \partial\Delta[m] & \xrightarrow{?} & Z' \\ \downarrow & & \downarrow \\ \Delta[m] & \xrightarrow{?} & X \end{array} \right\} \\ \downarrow & \lrcorner & \downarrow \\ Z_m = \left\{ \begin{array}{ccc} \partial\Delta[m] & \xrightarrow{?} & Z \\ \downarrow & & \downarrow \\ \Delta[m] & \xrightarrow{?} & Z \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{ccc} \partial\Delta[m] & \xrightarrow{?} & X \\ \downarrow & & \downarrow \\ \Delta[m] & \xrightarrow{?} & X \end{array} \right\} = X_m. \end{array}$$

When $m < n$, notice that

$$Z'_m \twoheadrightarrow \text{hom}(\partial\Delta[m] \rightarrow \Delta[m], Z' \rightarrow X), \quad Z_m \cong Z_m, \quad X_m \cong X_m; \quad (7)$$

then using Lemma 2.9 (in the case $L \cong A$ and $M \cong B$), we conclude that $Z_m \times_{X_m} Z_m \twoheadrightarrow \text{hom}(\partial\Delta[m] \rightarrow \Delta[m], Z' \times_X Z \rightarrow Z)$ is a cover in \mathcal{C} . When $m = n$, we only have to change the \twoheadrightarrow in (7) to \cong to obtain $Z_m \times_{X_m} Z_m \cong \text{hom}(\partial\Delta[m] \rightarrow \Delta[m], Z' \times_X Z \rightarrow Z)$. \square

Lemma 2.9. *Given a pull-back diagram in \mathcal{C} ,*

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A, \end{array}$$

covers $L \rightarrow A$, $M \rightarrow B$, $N \rightarrow L \times_A C$, and a morphism $M \rightarrow L$, then the natural map $M \times_L N \rightarrow B \times_A C$ is a cover. Moreover when $M \rightarrow B$ and $N \rightarrow L \times_A C$ are isomorphisms, $M \times_L N \rightarrow B \times_A C$ is an isomorphism.

Proof. We form the following pull-back diagram (where \bigcirc denotes unimportant pull-backs),

$$\begin{array}{ccccc}
 M \times_L N & \xrightarrow{\quad} & N & & \\
 \downarrow & \nearrow & \downarrow & \nearrow & \\
 \bigcirc & \xrightarrow{\quad} & \bigcirc & \xrightarrow{\quad} & C \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 M & \xrightarrow{\quad} & B & \xrightarrow{\quad} & A \\
 & \nearrow & \downarrow & \nearrow & \\
 & & L & \xrightarrow{\quad} & A
 \end{array}$$

$B \times_A C \xrightarrow{\quad} L \times_A C$
 $B \times_A L \xrightarrow{\quad} L$

Since M maps to both B and L , there exists a morphism $M \rightarrow B \times_A L$, fit into the diagram above. Since $L \twoheadrightarrow A$, all the objects in the diagram are representable in \mathcal{C} . Then the natural map $M \times_L N \rightarrow B \times_A C$ as a composition of covers is a cover itself. The statement on isomorphisms may be proven similarly. \square

2.2 Pull-back, generalized morphisms and various Morita equivalences

Let us first make the following observation: when $n = 1$ and \mathcal{C} is the category of Banach manifolds, hypercovers of n -groupoid objects give the concept of equivalence (or pull-back) of Lie groupoids. We explicitly study the case when $n = 2$: Let X be a 2-groupoid object in \mathcal{C} and let $Z_1 \rightrightarrows Z_0$ be in \mathcal{C} with structure maps as in (1) up to the level $m \leq 1$, and $f_m : Z_m \rightarrow X_m$ preserving the structure maps d_k^m and s_k^{m-1} for $m \leq 1$. Then $\text{hom}(\partial\Delta[m], Z)$ still makes sense for $m \leq 1$. We further suppose $f_0 : Z_0 \twoheadrightarrow X_0$ (hence $Z_0 \times_{X_0 \times X_0} X_1 \in \mathcal{C}$) and $Z_1 \twoheadrightarrow Z_0 \times_{X_0 \times X_0} X_1$. That is to say that the induced map from Z_m to the pull-back $\text{hom}(\partial\Delta[m] \rightarrow \Delta[m], Z \rightarrow X)$ is a cover for $m = 0, 1$. Then we form⁹

$$Z_2 = PB(\text{hom}(\partial\Delta[2], Z) \rightarrow \text{hom}(\partial\Delta[2], X) \leftarrow X_2).$$

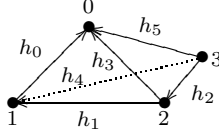
It is easy to see that the proof of Lemma 2.4 still guarantees $Z_2 \in \mathcal{C}$. Moreover there are $d_i^2 : Z_2 \rightarrow Z_1$ induced by the natural maps $\text{hom}(\partial\Delta[2], Z) \rightarrow Z_1$; $s_i^1 : Z_1 \rightarrow Z_2$ by

$$s_0^1(h) = (h, h, s_0^0(d_0^1(h)), s_0^1(f_1(h))), \quad s_1^1(h) = (s_0^0(d_1^1(h)), h, h, s_1^1(f_1(h)));$$

$m_i^Z : \Lambda(Z)_{3,i} \rightarrow Z_2$ via $m_i^X : \Lambda(X)_{3,i} \rightarrow X_2$ by for example

$$m_0^Z((h_2, h_5, h_3, \bar{\eta}_1), (h_4, h_5, h_0, \bar{\eta}_2), (h_1, h_3, h_0, \bar{\eta}_3)) = (h_2, h_4, h_1, m_0^X(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)),$$

and similarly for other m 's.



Then $Z_2 \rightrightarrows Z_1 \rightrightarrows Z_0$ is a 2-groupoid object in $(\mathcal{C}, \mathcal{T})$, and we call it the *pull-back 2-groupoid* by f . Moreover $f : Z \rightarrow X$ is a hypercover with f_0, f_1 and the natural map $f_2 : Z_2 \rightarrow X_2$.

⁹Strictly speaking, Z is not a simplicial object, but $\text{hom}(\partial\Delta[2], Z)$ as a fibre product of Z_1 's over Z_0 's still makes sense.

Definition 2.10. A *generalized morphism* between two n -groupoid objects X and Y in $(\mathcal{C}, \mathcal{T})$ consists of a *zig-zag* of strict maps $X \xleftarrow{\sim} Z \rightarrow Y$, where the map $Z \xrightarrow{\sim} X$ is a hypercover.

Proposition 2.11. A *composition of generalized morphisms is still a generalized morphism*.

Proof. This follows from Lemmas 2.6, 2.7 and 2.8. \square

Definition 2.12. Two n -groupoid objects X and Y in $(\mathcal{C}, \mathcal{T})$ are *Morita equivalent* if there is another n -groupoid object Z in $(\mathcal{C}, \mathcal{T})$ and maps $X \xleftarrow{\sim} Z \xrightarrow{\sim} Y$ such that both maps are hypercovers. By Lemmas 2.6, 2.7 and 2.8, this definition does give an equivalence relation. We call it *Morita equivalence* of n -groupoid objects in $(\mathcal{C}, \mathcal{T})$.

However, Morita equivalent Lie 2-groupoids correspond to Morita equivalent SLie groupoids [10]. Hence to obtain isomorphic stacky groupoid objects, we need a stricter equivalence relation.

Proposition-Definition 2.13. A strict map of n -groupoid objects $f : Z \rightarrow X$ is a *1-hypercover* if it is a hypercover with f_0 an isomorphism. Two n -groupoid objects X and Y in $(\mathcal{C}, \mathcal{T})$ are *1-Morita equivalent* if there is an n -groupoid object Z in $(\mathcal{C}, \mathcal{T})$ and maps $X \xleftarrow{\sim} Z \xrightarrow{\sim} Y$ such that both maps are 1-hypercovers. This gives an equivalence relation between n -groupoid objects, and we call it *1-Morita equivalence*.

Proof. It is easy to see that the composition of 1-hypercovers is still a 1-hypercover. We just have to notice that if both hypercovers $f : Z \rightarrow X$ and $f' : Z' \rightarrow X$ are 1-hypercovers, then the natural maps $Z_0 \leftarrow Z_0 \times_{X_0} Z'_0 \rightarrow Z'_0$ are isomorphisms since $Z_0 \times_{X_0} Z'_0 \cong Z_0 \cong X_0 \cong Z'_0$. \square

Remark 2.14. For a 1-hypercover $Z \rightarrow X$, since $f_0 : Z_0 \cong X_0$, we have $\text{hom}(\partial\Delta^1, Z) = \text{hom}(\partial\Delta^1, X)$. So the condition on f_1 in Def. 2.13 becomes $f_1 : Z_1 \twoheadrightarrow X_1$.

2.3 Cosk^m , Sk^m and finite data description

Often the conventional way with only finite layers of data to understand Lie group(oid)s is more conceptual in differential geometry. For a finite description of an n -groupoid, we introduce the functors Sk^m and Cosk^m from the category of simplicial objects in \mathcal{C} to itself [15, Section 2]. It is easy to describe Sk^m : $\text{Sk}^m(X)_k = X_k$ when $k \leq m$ and $\text{Sk}^m(X)_k$ only has degenerated simplices coming from X_m when $k > m$. Then Cosk^m is the right adjoint; that is,

$$\text{hom}(\text{Sk}^n(Y), X) \cong \text{hom}(Y, \text{Cosk}^n(X)).$$

In this section, we assume our category \mathcal{C} has a forgetful functor to the category of sets, which is the case for all our examples. Then we have

Proposition 2.15. *If X is an n -groupoid object in $(\mathcal{C}, \mathcal{T})$, then $\text{Cosk}^{n+1}(X) = X$.*

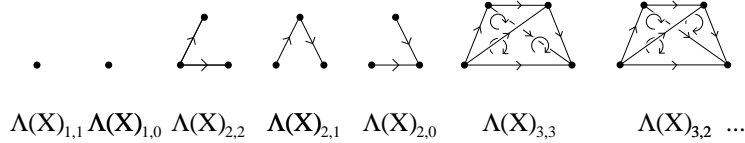
Proof. We only have to show this identity set-theoretically, which is shown in [15, Section 2]. \square

This tells us that it is possible to describe an n -groupoid object with only the first n layers and some extra data. The idea is to let $X_{n+1} := \Lambda[n+1, j](X)$, which is a certain fibre product involving $X_{k \leq n}$; then we produce X by

$$X = \text{Cosk}^{n+1} \text{Sk}^{n+1}(X_{n+1} \rightarrow X_n \rightarrow \cdots \rightarrow X_0). \quad (8)$$

When $n = 1$, this is a groupoid object in $(\mathcal{C}, \mathcal{T})$, as we have demonstrated in the introduction. Set-theoretically, these extra data are worked out when $n = 1, 2$ in [15]. We hereby work out the case of $n = 2$ in an enriched category $(\mathcal{C}, \mathcal{T})$, where representability in \mathcal{C} needs to be taken care of.

The extra data for a 2-groupoid object are associative “3-multiplications”. Following the notion of simplicial objects, we call d_i^m and s_j^m the face and degeneracy maps between X_i ’s, for $i = 0, 1, 2$; they still satisfy the coherence condition in (1). To simplify the notation and match it with the definition of groupoids, we use the notation \mathbf{t} for d_0^1 , \mathbf{s} for d_1^1 and e for s_0^0 . Then we can safely omit the upper indices for d_i^2 ’s and s_i^1 ’s. Actually we will omit the upper indices whenever it does not cause confusion. Similarly to the horn spaces $\text{hom}(\Lambda[m, j], X)$, given only these three layers, we define $\Lambda(X)_{m,j}$ to be the space of m elements in X_{m-1} glued along elements in X_{m-2} to a horn shape without the j -th face.



Here one imagines each j -dimensional face as an element in X_j . For example,

$$\begin{aligned} \Lambda(X)_{2,2} &= X_1 \times_{\mathbf{s}, X_0, \mathbf{s}} X_1, \quad \Lambda(X)_{2,1} = X_1 \times_{\mathbf{t}, X_0, \mathbf{s}} X_1, \quad \Lambda(X)_{2,0} = X_1 \times_{\mathbf{t}, X_0, \mathbf{t}} X_1, \\ &\dots, \quad \Lambda(X)_{3,0} = (X_2 \times_{d_2, X_1, d_1} X_2) \times_{d_1 \times d_2, \Lambda(X)_{2,0}, d_1 \times d_2} X_2. \end{aligned}$$

We remark that items (1a) and (1b) in the proposition-definition below imply that the $\Lambda(X)_{2,j}$ ’s and $\Lambda(X)_{3,j}$ ’s are representable in \mathcal{C} . Then with this condition we can define *3-multiplications* as morphisms $m_i : \Lambda(X)_{3,i} \rightarrow X_2$, $i = 0, \dots, 3$. With 3-multiplications, there are natural maps between $\Lambda(X)_{3,j}$ ’s. For example,

$$\Lambda(X)_{3,0} \rightarrow \Lambda(X)_{3,1}, \quad \text{by } (\eta_1, \eta_2, \eta_3) \rightarrow (m_0(\eta_1, \eta_2, \eta_3), \eta_2, \eta_3).$$

It is reasonable to ask them to be isomorphisms. In fact, set theoretically, this simply says that the following four equations are equivalent to each other:

$$\begin{aligned} \eta_0 &= m_0(\eta_1, \eta_2, \eta_3), & \eta_1 &= m_1(\eta_0, \eta_2, \eta_3), \\ \eta_2 &= m_2(\eta_0, \eta_1, \eta_3), & \eta_3 &= m_3(\eta_0, \eta_1, \eta_2). \end{aligned}$$

Proposition-Definition 2.16. A 2-groupoid object in $(\mathcal{C}, \mathcal{T})$ can be also described by three layers $X_2 \rightrightarrows X_1 \rightrightarrows X_0$ of objects in \mathcal{C} and the following data:

1. the face and degeneracy maps d_i^n and s_i^n satisfying (1) for $n = 1, 2$ as explained above, such that

- (a) [1-Kan] \mathbf{t} and \mathbf{s} are covers;
- (b) [2-Kan] $d_0 \times d_2 : X_2 \rightarrow \Lambda(X)_{2,1} = X_1 \times_{\mathbf{t}, X_0, \mathbf{s}} X_1$, $d_0 \times d_1 : X_2 \rightarrow \Lambda(X)_{2,2} = X_1 \times_{\mathbf{s}, X_0, \mathbf{s}} X_1$, and $d_1 \times d_2 : X_2 \rightarrow \Lambda(X)_{2,0} = X_1 \times_{\mathbf{t}, X_0, \mathbf{t}} X_1$ are covers.

2. morphisms (3-multiplications),

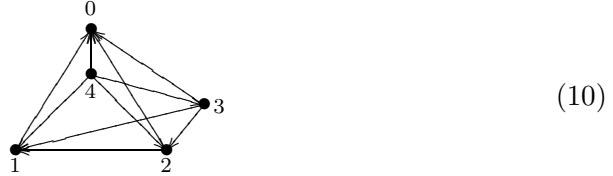
$$m_i : \Lambda(X)_{3,i} \rightarrow X_2, \quad i = 0, \dots, 3.$$

such that

- (a) the induced morphisms (by m_j as above) $\Lambda(X)_{3,i} \rightarrow \Lambda(X)_{3,j}$ are all isomorphisms;
- (b) the m_i 's are compatible with the face and degeneracy maps:

$$\begin{aligned} \eta &= m_1(\eta, s_0 \circ d_1(\eta), s_0 \circ d_2(\eta)) \text{ (which is equivalent to } \eta = m_0(\eta, s_0 \circ d_1(\eta), s_0 \circ d_2(\eta))), \\ \eta &= m_2(s_0 \circ d_0(\eta), \eta, s_1 \circ d_2(\eta)) \text{ (which is equivalent to } \eta = m_1(s_0 \circ d_0(\eta), \eta, s_1 \circ d_2(\eta))), \\ \eta &= m_3(s_1 \circ d_0(\eta), s_1 \circ d_1(\eta), \eta) \text{ (which is equivalent to } \eta = m_2(s_1 \circ d_0(\eta), s_1 \circ d_1(\eta), \eta)). \end{aligned} \tag{9}$$

- (c) the m_i 's are associative, that is, for a 4-simplex η_{01234} ,



if we are given faces η_{0i4} and η_{0ij} in X_2 , where $i, j \in \{1, 2, 3\}$, then the following two methods to determine the face η_{123} give the same element in X_2 :

- i. $\eta_{123} = m_0(\eta_{023}, \eta_{013}, \eta_{012})$;
- ii. we first obtain η_{ij4} 's using the m_i 's on the η_{0i4} 's; then we have

$$\eta_{123} = m_3(\eta_{234}, \eta_{134}, \eta_{124}).$$

Remark 2.17. Set-theoretically, this definition is that of [14]. In fact, it is enough to use one of the four multiplications m_j as therein, since one determines the others by item 2a. However, we use all the four multiplications here and later on in the proof to make it geometrically more direct. Here we see that this idea also applies well to, and even brings convenience to, other categories.

For example, in the case of a Lie 2-groupoid, i.e. when \mathcal{C} is the category of Banach manifolds with surjective submersions as covers, although the surjectivity of the maps in the 2-Kan condition (1b) insures the existence of the usual (2-) multiplication $m : X_1 \times_{\mathbf{t}, X_0, \mathbf{s}} X_1$ and inverse $i : X_1 \rightarrow X_1$ as explained in the introduction, these two maps are not necessarily continuous, or smooth, and m is not necessarily associative on the nose. For example, the Lie 2-groupoids coming from integrating Lie algebroids have two models [42]: the finite-dimensional one does not have a continuous 2-multiplication and the infinite-dimensional one has a smooth multiplication which does not satisfy associativity on the nose.

On the other hand, only having the usual 2-multiplication m and inverse map i , it is not guaranteed that the maps in the 2-Kan condition (1b) are submersions even when m and i are smooth. But being submersions is in turn very important to prove that $X_{n \geq 3}$ are smooth manifolds. Hence in the differential category, we cannot replace the 2-Kan condition by the usual 2-multiplication and inverse.

The nerve of $X_2 \rightrightarrows X_1 \rightrightarrows X_0$ To show that what we defined just now is the same as Def. 1.3, we form the *nerve* of a 2-groupoid $X_2 \rightrightarrows X_1 \rightrightarrows X_0$ in Prop-Def. 2.16. We first define

$$X_3 = \{(\eta_0, \eta_1, \eta_2, \eta_3) \mid \eta_0 = m_0(\eta_1, \eta_2, \eta_3), (\eta_1, \eta_2, \eta_3) \in \Lambda(X)_{3,0}\}.$$

Then $X_3 \cong \Lambda(X)_{3,0}$ is representable in \mathcal{C} . Moreover, we have the obvious face and degeneracy maps between X_3 and X_2 ,

$$\begin{aligned} d_i^3(\eta_0, \eta_1, \eta_2, \eta_3) &= \eta_i, \quad i = 0, \dots, 3 \\ s_0^2(\eta) &= (\eta, \eta, s_0 \circ d_1(\eta), s_0 \circ d_2(\eta)), \\ s_1^2(\eta) &= (s_0 \circ d_0(\eta), \eta, \eta, s_1 \circ d_2(\eta)), \\ s_2^2(\eta) &= (s_1 \circ d_0(\eta), s_1 \circ d_1(\eta), \eta, \eta). \end{aligned}$$

The coherency (9) insures that $s_i^2(\eta) \in X_3$. It is also not hard to see that these maps together with the $d_i^{\leq 2}$'s and $s_i^{\leq 1}$'s satisfy (1) for $n \leq 3$.

Then the nerve can be easily described by (8). More concretely, X_m is made up of those m -simplices whose 2-faces are elements of X_2 and such that each set of four 2-faces gluing together as a 3-simplex is an element of X_3 . That is,

$$X_m = \{f \in \text{hom}_2(sk_2(\Delta_m), X_2) \mid f \circ (d_0 \times d_1 \times d_2 \times d_3)(sk_3(\Delta_m)) \subset X_3\},$$

where hom_2 denotes the homomorphisms restricted to the 0, 1, 2 level and X_2 is understood as the tower $X_2 \rightrightarrows X_1 \rightrightarrows X_0$ with all degeneracy and face maps. Then there are obvious face and degeneracy maps which naturally satisfy (1).

However what is nontrivial is that the associativity of the m_i 's assures that X_m is representable in \mathcal{C} . We prove this by an inductive argument. Let $S_j[m]$ be the contractible simplicial set whose sub-faces all contain the vertex j and whose only non-degenerate faces are of dimensions 0, 1 and 2. Then similarly to [21, Lemma 2.4], we now show that $\text{hom}_2(S_j[m], X_2)$ is representable in \mathcal{C} . Since $S_j[m]$ is constructed by adding 0, 1, 2-dimensional faces, it is formed by the procedure

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \uparrow & & \uparrow \\ \Lambda[m, j] & \longrightarrow & \Delta[m] \end{array}$$

with $m \leq 2$. The dual pull-back diagram shows that $\text{hom}_2(S_j[m], X_2)$ is representable by induction

$$\begin{array}{ccc} \text{hom}_2(S', X_2) & \longleftarrow & \text{hom}_2(S, X_2) \\ \downarrow & & \downarrow \\ \text{hom}_2(\Lambda[m, j], X_2) & \longleftarrow & \text{hom}_2(\Delta[m], X_2), \end{array}$$

since $\text{hom}_2(\Delta[m], X_2) \rightarrow \text{hom}_2(\Lambda[m, j], X_2)$ are covers by items 1a and 1b in the Prop-Def 2.16.

Next we use induction to show that $X_m = \text{hom}_2(S_0[m], X_2)$. Similarly we will have $X_m = \text{hom}_2(S_j[m], X_2)$. It is clear that $\tilde{f} \in X_m$ restricts to $\tilde{f}|_{S_0[m]} \in \text{hom}_2(S_0[m], X_2)$. We only have to show that $f \in \text{hom}_2(S_0[m], X_2)$ extends uniquely to $\tilde{f} \in X_m$. It is certainly true for $n = 0, 1, 2, 3$ just by definition. Suppose $X_{m-1} = \text{hom}_2(S_0[m-1], X_2)$. Then to get $f \in \text{hom}_2(S_0[m], X_2)$ from $f' \in \text{hom}_2(S_0[m-1], X_2)$, we add a new point m and $(m-1)$ new faces $(0, i, m)$, $i \in \{1, 2, \dots, m-1\}$ and dye them red¹⁰. Using 3-multiplication m_0 , we can determine face (i, j, m) by $(0, i, m)$, $(0, j, m)$ and $(0, i, j)$ and dye these newly decided faces blue. Now we want to see that each of the four faces attached together are in X_3 ; then f is extended to $\tilde{f} \in X_m$. We consider various cases:

1. if none of the four faces contains the vertex m , then by the induction condition, they are in X_3 .
2. if one of the four faces contains m , then there are three faces containing m ; we again have two sub-cases:
 - (a) if those three contain only one blue face of the form (i, j, m) , $i, j \in \{1, \dots, (m-1)\}$, then the four faces must contain three red faces and one blue face. According to our construction, these four faces are in X_3 ;
 - (b) if those three contain more than one blue face, then they must contain exactly three blue faces. Then according to associativity (inside the 5-gon $(0, i, j, k, m)$), these four faces are also in X_3 .

Now we finish the induction, hence X_m is representable in \mathcal{C} and it is determined by the first three layers. Similarly we can prove $\text{hom}(\Lambda[m, j], X) = \text{hom}_2(S_0[m], X_2)$. Hence $\text{hom}(\Lambda[m, j], X) = X_m$, and we finish the proof of the Prop-Def. 2.16, which is summarized in the following two lemmas:

Lemma 2.18. *The nerve X of a 2-groupoid object $X_2 \Rightarrow X_1 \Rightarrow X_0$ in $(\mathcal{C}, \mathcal{T})$ as in Prop-Def. 2.16 is a 2-groupoid object in $(\mathcal{C}, \mathcal{T})$ as in Def. 1.3.*

Lemma 2.19. *The first three layers of a 2-groupoid object in $(\mathcal{C}, \mathcal{T})$ as in Def. 1.3 is a 2-groupoid object in $(\mathcal{C}, \mathcal{T})$ as in Prop-Def. 2.16.*

Proof. The proof is more complicated and similar to the case of 1-groupoids in the introduction. Here we point out that the 3-multiplications m_j are given by $\text{Kan}!(3, j)$ and the associativity is given by $\text{Kan}!(3, 0)$ and $\text{Kan}(4, 0)$. \square

3 Stacky groupoids in various categories

Given a category \mathcal{C} with a singleton Grothendieck pretopology \mathcal{T} (not necessarily satisfying Assumptions 2.1), we can develop the theory of stacks [1]. The Yoneda lemma also holds here; namely we can embed \mathcal{C} into the 2-category of stacks built upon $(\mathcal{C}, \mathcal{T})$. We call such stacks *representable stacks*. Moreover, weaker than this, a kind of nice stack, which

¹⁰More precisely, they are the image of these under the map f .

we call a *presentable stack*, corresponds to the groupoid objects in \mathcal{C} . For this one needs another singleton Grothendieck pretopology \mathcal{T}' .

The theory of presentable stacks in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ (see Table 2) has been developed over the past few decades in the algebraic category, where they are known as Delign–Munford (DM) stacks and Artin stacks in the étale and general cases respectively (see for example [38] for a good summary), and recently in the differential category by [6, 24, 31] and [27] (in the context of orbifolds) and the topological category by [17, 24, 29]. We refer the reader to these references for these concepts and only sketch the idea here.

First, to distinguish, we call a cover in topology \mathcal{T}' a *projection*. We call a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ between stacks in $(\mathcal{C}, \mathcal{T})$ a *representable projection* if for every map $U \rightarrow \mathcal{Y}$ for $U \in \mathcal{C}$, the pull-back map $\mathcal{X} \times_U \mathcal{Y} \rightarrow U$ is a projection in \mathcal{C} (this implies that $\mathcal{X} \times_U \mathcal{Y}$ is representable in \mathcal{C}). A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ between stacks in $(\mathcal{C}, \mathcal{T})$ is an *epimorphism* if for every $U \rightarrow \mathcal{Y}$ with $U \in \mathcal{C}$, there exists a cover $V \rightarrow U$ in \mathcal{T} fit in the following 2-commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\exists} & U \\ \downarrow & & \downarrow \forall \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Then a *presentable stack*¹¹ in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ is a stack \mathcal{X} which possesses a *chart* $X \in \mathcal{C}$ such that $X \rightarrow \mathcal{X}$ is a representable projection and an epimorphism w.r.t. \mathcal{T} .

To define the 2-category of groupoids in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$, we need to define first a *surjective projection* between presentable stacks. We adopt the definition in [36, Section 3], which is that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a *projection* if $X \times_{\mathcal{Y}} Y \rightarrow Y$ is a projection where X and Y are charts of \mathcal{X} and \mathcal{Y} respectively. f is further a *surjective projection* if it is an epimorphism of stacks. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a surjective projection from a presentable stack to an object in \mathcal{C} , then the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ for any map $\mathcal{Z} \rightarrow \mathcal{Y}$ is again a presentable stack. Then a *groupoid object* in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ is as we imagine: $G := G_1 \rightrightarrows G_0$ with $G_i \in \mathcal{C}$, all the groupoid structure maps morphisms in \mathcal{C} , and source and target maps surjective projections. One subtle point is that for a principal G bundle X over S , the map $X \rightarrow S$ has to be a surjective projection.

The upshot of this theory is that the 2-category of presentable stacks is equivalent to the 2-category of groupoids¹² in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$. This implies that a presentable stack is presented by a groupoid object (which may not be unique), and a morphism between presentable stacks is presented by an H.S. bibundle. There is also a correspondence on the level of 2-morphisms.

Moreover, we have

Lemma 3.1. *If \mathcal{T}' satisfies Assumptions 2.1 with terminal object $*'$ and any map $X \rightarrow *'$ is an epimorphism w.r.t. \mathcal{T} for all $X \in \mathcal{C}$, then surjective projections serve as covers of a certain singleton Grothendieck pretopology \mathcal{T}'' which satisfies Assumptions 2.1.*

Proof. The only thing which is not obvious is to check that if $Z \rightarrow X$ is an epimorphism in $(\mathcal{C}, \mathcal{T})$ and $Y \rightarrow X$ is a morphism in \mathcal{C} , then the pull-back $Y \times_X Z \rightarrow Y$ is still an

¹¹This is a slightly different set-up to that in the usual references, but it says exactly the same thing by [6, Lemma 2.13].

¹²In the algebraic category, usually we need more conditions for such a groupoid to present an algebraic stack. For example, $(\mathbf{t}, \mathbf{s}) : G_1 \rightarrow G_0 \times G_0$ is separated and quasi-compact [22, Prop. 4.3.1]. But in the differential and topological categories, we do not require extra conditions. See Table 2.

epimorphism in $(\mathcal{C}, \mathcal{T})$, for $X, Y, Z \in \mathcal{C}$. For any $U \rightarrow Y$, we have a composed morphism $U \rightarrow X$. Since $Z \rightarrow X$ is an epimorphism, there exists a cover $V \rightarrow U$ in \mathcal{T} , such that the rectangle diagram in the diagram below commutes

$$\begin{array}{ccc}
 V & \longrightarrow & U \\
 \downarrow & & \downarrow \\
 Z \times_X Y & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 Z & \longrightarrow & X
 \end{array}$$

(A curved arrow points from V to Z on the left side of the diagram.)

Hence there exists a morphism $V \rightarrow Z \times_X Y$ such that the up-level small square commutes; that is, $Z \times_X Y \rightarrow Y$ is an epimorphism. \square

Therefore by definition, we have

Corollary 3.2. *The groupoid objects and H.S. morphisms in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ are exactly the same as the (1-) groupoid objects and H.S. morphisms in $(\mathcal{C}, \mathcal{T}'')$.*

We list possible $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ and \mathcal{T}'' with their theory of presentable stacks in Table 2.

Table 2: Example of categories for a theory of presentable stacks

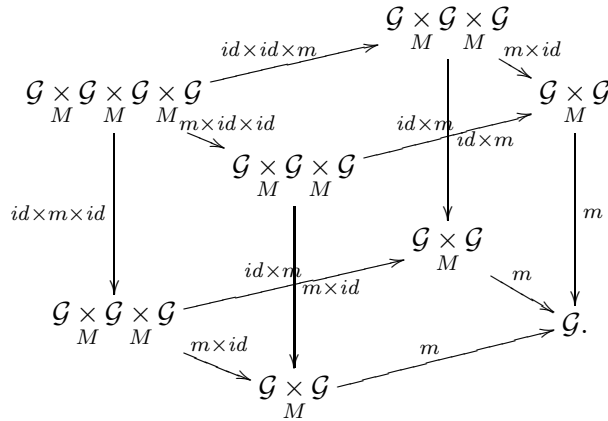
Theory of	$(\mathcal{C}, \mathcal{T}, \mathcal{T}')$	covers of \mathcal{T}''	presented by
Differentiable stacks	$(\mathcal{C}_1, \mathcal{T}_1, \mathcal{T}'_1)$	same as \mathcal{T}'_1	Lie groupoids
Topological stacks	$(\mathcal{C}_2, \mathcal{T}_2, \mathcal{T}'_2)$	surjective maps with local sections	topological groupoids
Artin stacks	$(\mathcal{C}_3, \mathcal{T}_3, \mathcal{T}'_3)$	same as \mathcal{T}'_3	groupoid schemes with extra conditions

From now on, we restrict ourselves to only the first two situations described in Table 2; that is, when we mention $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ and \mathcal{T}'' , it is either $(\mathcal{C}_1, \mathcal{T}_1, \mathcal{T}'_1)$ and \mathcal{T}''_1 or $(\mathcal{C}_2, \mathcal{T}_2, \mathcal{T}'_2)$ and \mathcal{T}''_2 .

Remark 3.3. The definition of a groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ is the same as a groupoid object in $(\mathcal{C}, \mathcal{T}')$ even though we have to use \mathcal{T} to define epimorphisms. For example, Lie groupoids are the groupoid objects in $(\mathcal{C}_1, \mathcal{T}_1, \mathcal{T}'_1)$ and also the groupoid objects in $(\mathcal{C}_1, \mathcal{T}_1)$, since both require the source and target to be surjective submersions. Topological groupoids are the groupoid objects in $(\mathcal{C}_2, \mathcal{T}_2, \mathcal{T}'_2)$ requiring source and target to be surjective maps with local sections. But with the identity section, the conditions for the source and target naturally hold. Hence topological groupoids are also the groupoid objects in $(\mathcal{C}_2, \mathcal{T}'_2)$. However the definition of H.S. morphisms in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ and $(\mathcal{C}, \mathcal{T}')$ is not necessarily the same. Hence when the condition in Lemma 3.1 is satisfied, the definition of n -groupoid in $(\mathcal{C}, \mathcal{T}'')$ and $(\mathcal{C}, \mathcal{T}')$ is not necessarily the same.

Definition 3.4 (stacky groupoid). A stacky groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ over an object $M \in \mathcal{C}$ consists of the following data:

1. a presentable stack \mathcal{G} ;
2. (source and target) maps $\bar{s}, \bar{t} : \mathcal{G} \rightarrow M$ which are surjective projections;
3. (multiplication) a map $m : \mathcal{G} \times_{\bar{s}, M, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}$, satisfying the following properties:
 - (a) $\bar{t} \circ m = \bar{t} \circ pr_1$, $\bar{s} \circ m = \bar{s} \circ pr_2$, where $pr_i : \mathcal{G} \times_{\bar{s}, M, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}$ is the i -th projection map $\mathcal{G} \times_{\bar{s}, M, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}$;
 - (b) associativity up to a 2-morphism; i.e., there is a 2-morphism a between maps $m \circ (m \times id)$ and $m \circ (id \times m)$;
 - (c) the 2-morphism a satisfies a higher coherence described as follows: let the 2-morphisms on the each face of the cube be a_i ¹³ arranged in the following way: front face (the one with the most \mathcal{G} 's) a_1 , back a_5 ; up a_4 , down a_2 ; left a_6 , right a_3 ;



We require

$$(a_6 \times id) \circ (id \times a_2) \circ (a_1 \times id) = (id \times a_5) \circ (a_4 \times id) \circ (id \times a_3).$$

4. (identity section) a morphism $\bar{e}: M \rightarrow \mathcal{G}$ such that

- (a) the identities

$$m \circ ((\bar{e} \circ \bar{\mathbf{t}}) \times id) \xRightarrow{b_l} id, \quad m \circ (id \times (\bar{e} \circ \bar{\mathbf{s}})) \xRightarrow{b_r} id,$$

hold¹⁴ up to 2-morphisms b_l and b_r . Or equivalently there are two 2-morphisms

$$\begin{array}{ccc} m \circ (id \times \bar{e}) \xrightarrow{b_r} pr_1 : \mathcal{G} \times_{\bar{s}, M} M \rightarrow \mathcal{G}, & m \circ (\bar{e} \times id) \xrightarrow{b_l} pr_2 : M \times_{M, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}, \\ g\bar{e}(y) \rightarrow g & & \bar{e}(x)g \rightarrow g \end{array}$$

where $y = \bar{\mathbf{s}}(g)$ and $x = \bar{\mathbf{t}}(g)$.

- (b) the restriction of b_r and b_l on $m \circ (\bar{e} \times \bar{e}) \xrightarrow[b_r]{b_l} \bar{e}$ are the same;

¹³All the a_i 's are generated by a , except that a_4 is *id*.

¹⁴In particular, by combining with the surjectivity of \bar{s} and \bar{t} , one has $\bar{s} \circ \bar{e} = id$, $\bar{t} \circ \bar{e} = id$ on M . In fact if $x = \bar{t}(g)$, then $\bar{e}(x) \cdot g \sim g$ and $\bar{t} \circ m = \bar{t} \circ pr_1$ imply that $\bar{t}(\bar{e}(x)) = \bar{t}(g) = x$.

(c) the composed 2-morphism below, with $y = \bar{s}(g_2)$,

$$g_1 g_2 \xrightarrow{b_r^{-1}} (g_1 g_2) \bar{e}(y) \xrightarrow{a} g_1(g_2 \bar{e}(y)) \xrightarrow{b_r} g_1 g_2$$

is the identity;¹⁵

(d) similarly with $x = \bar{t}(g_1)$,

$$g_1 g_2 \xrightarrow{b_l^{-1}} \bar{e}(x)(g_1 g_2) \xrightarrow{a^{-1}} (\bar{e}(x) g_1) g_2 \xrightarrow{b_l} g_1 g_2$$

is the identity;

(e) with $x = \bar{s}(g_1) = \bar{t}(g_2)$,

$$g_1 g_2 \xrightarrow{b_l^{-1}} (g_1 \bar{e}(x)) g_2 \xrightarrow{a} g_1(\bar{e}(x) g_2) \xrightarrow{b_r} g_1 g_2,$$

is the identity.

5. (inverse) an isomorphism of stacks $\bar{i} : \mathcal{G} \rightarrow \mathcal{G}$ such that, the following identities

$$m \circ (\bar{i} \times id \circ \Delta) \Rightarrow \bar{e} \circ \bar{s}, \quad m \circ (id \times \bar{i} \circ \Delta) \Rightarrow \bar{e} \circ \bar{t},$$

hold up to 2-morphisms, where Δ is the diagonal map: $\mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$.

We are specially interested in the differential category.

Definition 3.5. When $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ is the differential category $(\mathcal{C}_1, T_1, T'_1)$, we call a stacky groupoid object $\mathcal{G} \Rightarrow M$ a *stacky Lie groupoid* (*SLie groupoid* for short). When \mathcal{G} is furthermore an étale differentiable stack and the identity e is an immersion of differentiable stacks, we call it a *Weinstein groupoid* (*W-groupoid* for short).

¹⁵We can also state this without any reference to objects. We notice that $pr_1 \circ (m \times id)$ and $m \circ (pr_1 \times pr_2)$ are the same map from $\mathcal{G} \times_M \mathcal{G} \times_M M$ to \mathcal{G} , but as the diagram indicates,

$$\begin{array}{ccc} \mathcal{G} \times_M \mathcal{G} \times_M M & \xrightarrow{m \times id} & \mathcal{G} \times_M M \\ \downarrow \text{id} \times (m \circ (id \times \bar{e})) \quad \downarrow pr_1 \times pr_2 & & \downarrow pr_1 \quad \downarrow m \circ (id \times \bar{e}) \\ \mathcal{G} \times_M \mathcal{G} & \xrightarrow{m} & \mathcal{G}, \end{array} \quad (11)$$

they are related also via a sequence of 2-morphisms:

$$pr_1 \circ (m \times id) \xrightarrow{b_r^{-1} \odot id} m \circ (id \times \bar{e}) \circ (m \times id) \xrightarrow{a} m \circ (id \times (m \circ (id \times \bar{e}))) \xrightarrow{id \odot (id \times b_r)} m \circ (pr_1 \times pr_2), \quad (12)$$

where \odot denotes conjunction of 2-morphisms, so that for example $b_r^{-1} : pr_1 \rightarrow m \circ (id \times \bar{e})$ is a 2-morphism, $id : m \times id \rightarrow m \times id$ is a 2-morphism, and the conjunction $b_r^{-1} \odot id$ gives a 2-morphism between the composed morphisms $pr_1 \circ (m \times id) \xrightarrow{b_r^{-1} \odot id} m \circ (id \times \bar{e}) \circ (m \times id)$. We require that the composed 2-morphisms be id , that is that

$$(id \odot (id \times b_r)) \circ a \circ (b_r^{-1} \odot id) = id,$$

where \circ is simply the composition of 2-morphisms.

Remark 3.6. This definition of W-groupoid is different from the one in [36]: here we add various higher coherences on 2-morphisms which make the definition stricter but still allow the W-groupoids $\mathcal{G}(A)$ and $\mathcal{H}(A)$, which are the integration objects of the Lie algebroid A constructed in [36]. Hence we remove

“Moreover, restricting to the identity section, the above 2-morphisms between maps are the *id* 2-morphisms. Namely, for example, the 2-morphism α induces the *id* 2-morphism between the following two maps:

$$m \circ ((m \circ (\bar{e} \times \bar{e} \circ \delta)) \times \bar{e} \circ \delta) = m \circ (\bar{e} \times (m \circ (\bar{e} \times \bar{e} \circ \delta)) \circ \delta),$$

where δ is the diagonal map: $M \rightarrow M \times M$.”

since it is implied by item 4b and item 4c.

On the other hand, we do not add higher coherences for the 2-morphisms involving the inverse map. This is because we can always find c'_r and c'_l that satisfy correct higher coherence conditions, possibly with a modified inverse map. See Section 3.2.

With some patience, we can check that the list of coherences on 2-morphisms given here generates all the possible coherences on these 2-morphisms. In fact, item 4c and item 4d are redundant (see [23, Chapter VII.1]). But we list them here since it makes more convenient for us to use later. We also notice that the cube condition (3c) is the same as the pentagon condition

$$[((gh)k)l \rightarrow (g(hk))l \rightarrow g((hk)l) \rightarrow g(h(kl))] = [((gh)k)l \rightarrow (gh)(kl) \rightarrow g(h(kl))].$$

3.1 Good charts

Given a stacky groupoid $\mathcal{G} \Rightarrow M$ in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$, the identity map $\bar{e} : M \rightarrow \mathcal{G}$ corresponds to an H.S. morphism from $M \Rightarrow M$ to $G_1 \Rightarrow G_0$ for some presentation of \mathcal{G} . But it is not clear whether M embeds into G_0 . It is not even obvious whether there is a map $M \rightarrow G_0$. In general, one could ask: if there is a map from an $M \in \mathcal{C}$ to a presentable stack \mathcal{G} , when can one find a chart G_0 of \mathcal{G} such that $M \rightarrow \mathcal{G}$ lifts to $M \rightarrow G_0$, namely when is the H.S. morphism $M \Rightarrow M$ to $G_1 \Rightarrow G_0$ a strict groupoid morphism? If the stack \mathcal{G} is étale, can we find an étale chart G_0 ? If such G_0 exists, we call it a *good chart* or *good étale chart* if it is furthermore étale, and we call $G_1 \Rightarrow G_0$ a *good groupoid presentation* for the map $M \rightarrow \mathcal{G}$.

We show the existence of good (étale) charts in the differential category $(\mathcal{C}_1, \mathcal{T}_1, \mathcal{T}'_1)$ by the following lemmas. It turns out that the étale case is easier and when $M \rightarrow \mathcal{G}$ is an immersion we can always achieve an étale chart.

Lemma 3.7. *For an immersion $\bar{e} : M \rightarrow \mathcal{G}$ from a manifold M to an étale stack \mathcal{G} , there is an étale chart G_0 of \mathcal{G} such that \bar{e} lifts to an embedding $e : M \rightarrow G_0$.*

Proof. Take an arbitrary étale chart G_0 of \mathcal{G} . The idea is to find an “open neighborhood” U of M in \mathcal{G} with the property that M embeds in U and there is an étale representable map $U \rightarrow \mathcal{G}$. Since $G_0 \rightarrow \mathcal{G}$ is an étale chart, in particular epimorphic, $G_0 \sqcup U \rightarrow \mathcal{G}$ is an étale representable epimorphism¹⁶, that is, a new étale chart of \mathcal{G} . Then the lemma is proven since $M \hookrightarrow G_0 \sqcup U$ is an embedding.

Now we look for such a U . Since $\bar{e} : M \rightarrow \mathcal{G}$ is an immersion, the pull-back $M \times_{\mathcal{G}} G_0 \rightarrow G_0$ is an immersion and $M \times_{\mathcal{G}} G_0 \rightarrow M$ is an étale epimorphism. We cover M by small enough open charts V_i so that each V_i lifts to an isomorphic open chart V'_i on $M \times_{\mathcal{G}} G_0$. Then $V'_i \rightarrow G_0$ is an immersion, so locally it is an embedding. Therefore we can divide V_i

¹⁶Note that being étale implies being submersive.

into even smaller open charts V_{ij} such that $V_{ij} \cong V'_{ij} \rightarrow G_0$ is an embedding. Hence we might assume that the V_i 's form an open covering of M such that \bar{e} lifts to embeddings $e_i : V_i \hookrightarrow G_0$. This appears in the language of Hilsun–Skandalis (H.S.) bibundles as the diagram on the right:

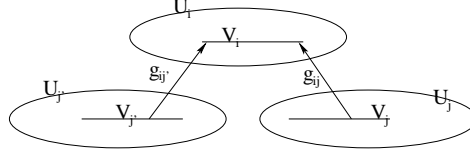
$$\begin{array}{ccccc}
V'_i \subset & M \times_{\mathcal{G}} G_0 & \longrightarrow & G_0 & \\
\downarrow & \downarrow & & \downarrow & \\
V_i \subset & M & \longrightarrow & \mathcal{G} & \\
& \downarrow & & \downarrow & \\
& M \supset V_i & & &
\end{array}
\quad
\begin{array}{ccc}
& & V'_i \subset M \times_{\mathcal{G}} G_0 \\
& \nearrow J_l & \searrow J_r \\
& & G_0
\end{array}$$

$\begin{array}{c} M \\ \parallel \\ M \\ \downarrow \\ M \supset V_i \end{array}$
 $\begin{array}{c} G_1 \\ \parallel \\ G_0 \end{array}$

Here $e_i = J_r \circ \sigma_i$. Since the action of G_1 on the H.S. bibundle is free and transitive, there exists a unique groupoid bisection g_{ij} such that $e_i \cdot g_{ij} = e_j$ on the overlap $V_i \cap V_j$. Since $G_1 \Rightarrow G_0$ is étale, the bisection g_{ij} extends uniquely to \bar{g}_{ij} on an open set $\bar{U}_{ij} \subset G_1$. Moreover, there exist open sets $U_i \supset e_i(V_i)$ of G_0 such that

$$e_i(V_i \cap V_j) \subset \mathbf{t}_G(\bar{U}_{ij}) =: U_{ji} \subset U_i, \quad e_j(V_i \cap V_j) \subset \mathbf{s}_G(\bar{U}_{ij}) =: U_{ij} \subset U_j.$$

Since $e_j \cdot g_{ij}^{-1} = e_i$, which implies that $g_{ji} = g_{ij}^{-1}$, these sets are well-defined.



Because of uniqueness and because $g_{ij} \cdot g_{jk} = g_{ik}$, we have $\bar{g}_{ij} \cdot \bar{g}_{jk} = \bar{g}_{ik}$ on the open subsets $\bar{U}_{ijk} := \{(\bar{g}_{ij}, \bar{g}_{jk}, \bar{g}_{ik}) : \bar{g}_{ij} \cdot \bar{g}_{jk} \text{ exists and is in } \bar{U}_{ik}\}$. Then

$$e_i(V_i \cap V_j \cap V_k) \subset U_{ijk} := \mathbf{t}_G(\text{Im}(\bar{U}_{ijk} \rightarrow \bar{U}_{ij})) \subset U_{ji} \cap U_{ki} \subset U_i,$$

and similarly for j and k . Therefore with these U 's we are in the situation of a *germ of manifolds* of M defined as below.

A *germ of manifolds at a point m* is a series of manifolds U_i containing the point m such that each U_i agrees with U_j in a smaller open set $(m \in) U_{ji} \subset U_i$ by $x \sim f_{ji}(x)$, with $f_{ji} : U_i \rightarrow U_j$ satisfying the cocycle condition $f_{kj} \circ f_{ji} = f_{ki}$. A *compatible riemannian metric* of a germ of manifolds consists of a riemannian metric g^i on each U_i such that two such riemannian metrics g^i and g^j on U_i and U_j agree with each other in the sense that $g^i(x) = g^j(f_{ji}(x))$ in a smaller open set (possibly a subset of U_{ji}). With this, one can define the exponential map \exp at m using the usual exponential map of a riemannian manifold, provided the germ is finite, meaning that there are finitely many manifolds in the germ (which is true in our case, since V_i intersects finitely many other V_j 's). Then \exp gives a Hausdorff manifold containing m .

If a series of locally finite manifolds U_i and morphisms f_{ji} form a germ of manifolds for every point of a manifold M , we call it a *germ of manifolds of M* . Here local finiteness means that any open set in M is contained in finitely many U_i 's and M has the topology induced by the U_i 's, that is that $M \cap U_i$ is open in M . We can always endow each of these with a compatible riemannian metric, beginning with any riemannian metric g^i on U_i and

modifying it to the sum $g^i(x) := \sum_{k, x \in U_{ki}} g^k(f_{ki}(x))$ (with $f_{ii}(x) = x$) at each point $x \in U_i$. In this situation, one can take a tubular neighborhood U of M by the *exp* map of the germ. Then U is a Hausdorff manifold.

Applying the above construction to our situation, we have a Hausdorff manifold $U \supset M$ with the same dimension as G_0 . U is basically glued by small enough open subsets $\tilde{U}_i = U \cap U_i$ containing the V_i 's along $\tilde{U}_{ij} := U \cap U_{ij}$ so that the gluing result U is still a Hausdorff manifold. Therefore U is presented by $\sqcup \tilde{U}_{ij} \rightrightarrows \sqcup \tilde{U}_i$, which maps to $G_1 \rightrightarrows G_0$ via $U_{ij} \cong \tilde{U}_{ij} \hookrightarrow G_1$. So there is a map $\pi : U \rightarrow \mathcal{G}$. Since the $\tilde{U}_i \rightarrow G_0$ are étale maps, by the technical lemma below, π is a representable étale map. \square

Lemma 3.8. *Given a manifold X and an (étale) differentiable stack \mathcal{Y} , a map $f : X \rightarrow \mathcal{Y}$ is an (étale) representable submersion if and only if there exists an (étale) chart Y_0 of \mathcal{Y} such that the induced local maps $X_i \rightarrow Y_0$ are (étale) submersions, where $\{X_i\}$ is an open covering of X .*

Proof. For any $V \rightarrow \mathcal{Y}$, $X_i \times_{\mathcal{Y}} V = X_i \times_{Y_0} Y_0 \times_{\mathcal{Y}} V$ is representable and $X_i \times_{\mathcal{Y}} V \rightarrow V$ is an (étale) submersion since $X_i \rightarrow Y_0$ and $Y_0 \rightarrow \mathcal{Y}$ are representable (étale) submersions. Since the X_i 's glue together to X , the $X_i \times_{\mathcal{Y}} V$ with the inherited gluing maps glue to a manifold $X \times_{\mathcal{Y}} V$. Since being an (étale) submersion is a local property, $X \times_{\mathcal{Y}} V \rightarrow V$ is an (étale) submersion.

$$\begin{array}{ccc}
 X_i \times_{\mathcal{Y}} V & \searrow & \\
 \downarrow & \curvearrowright & X_k \times_{\mathcal{Y}} V \\
 X_j \times_{\mathcal{Y}} V & \swarrow &
 \end{array}
 \quad \Leftrightarrow \quad
 \begin{array}{ccc}
 X_i & \searrow & \\
 \downarrow & \curvearrowright & X_k \\
 X_j & \swarrow &
 \end{array}$$

\square

Remark 3.9. If \bar{e} is the identity map of a W-groupoid $\mathcal{G} \rightrightarrows M$, then an open neighborhood of M in U has an induced local groupoid structure from the stacky groupoid structure [36, Section 5].

We further prove the same lemma in the non-étale case.

Lemma 3.10. *For a morphism $\bar{e} : M \rightarrow \mathcal{G}$ from a manifold M to a differentiable stack \mathcal{G} , there is a chart G_0 of \mathcal{G} such that \bar{e} lifts to an embedding $e : M \rightarrow G_0$.*

Proof. We follow the proof of the étale case, but replace “étale map” with “submersion”. We need a U with a representable submersion to \mathcal{G} and an embedding of M into U . There are two differences: first, V_i embeds in V_i' instead of being isomorphic to it, and we do not have an embedding $V_i' \hookrightarrow G_0$; second, since $G_1 \rightrightarrows G_0$ is not étale, the bisection g_{ij} does not extend uniquely to some \bar{g}_{ij} and we cannot have the cocycle condition immediately.

The first difference is easy to compensate for: given any morphism $f : N_1 \rightarrow N_2$, we can always view it as a composition of an embedding and a submersion as $N_1 \xrightarrow{id \times f} N_1 \times N_2 \xrightarrow{pr_2} N_2$. In our case, we have the decomposition $M \times_{\mathcal{G}} G_0 \hookrightarrow H_0 \rightrightarrows G_0$; then we use the pull-back groupoid $H_1 := G_1 \times_{G_0 \times G_0} H_0 \times H_0$ over H_0 to replace G . Thus we obtain an embedding

$V'_i \rightarrow H_0$ and so an embedding $V_i \rightarrow H_0$. Then since $H_1 \Rightarrow H_0$ is Morita equivalent to $G_1 \Rightarrow G_0$, we just have to replace G by H or call H our new G . It was not possible to do so in the étale case since H_0 might not be an étale chart of \mathcal{G} .

For the second difference, first of all we could assume M to be connected to construct such a U . Otherwise we take the disjoint union of such U 's for each connected component of M .

Then take any V_i and consider all the charts V_j intersecting V_i . We choose \bar{g}_{ij} extending g_{ij} on an open set \bar{U}_{ij} . As before we define the open sets U_i , the U_j 's and the U_{ij} 's. Then for V_j and $V_{j'}$ both intersecting V_i , we choose $\bar{g}_{jj'}$ to be the one extending (see below) $\bar{g}_{ij}^{-1}\bar{g}_{ij'}$ with $\mathbf{s}_G(\bar{g}_{ij}^{-1}\bar{g}_{ij'})$ in the triple intersection $\bar{g}_{ij}^{-1} \cdot (\bar{g}_{ij} \cdot U_j) \cap U_{j'}$, where multiplication applies when it can. Since the \bar{g} 's are local bisections, $\bar{g} \cdot -$ is an isomorphism. Identifying via these isomorphisms, we view and denote the above intersection as $U_{j'ij}$ for simplicity.

Now we clarify in which sense and why the extension always exists. Let us assume $\dim M = m$, $\dim G_i = n_i$. Here we identify each V_j with its embedded image in G_0 and require every V_j to be relatively closed in U_j . Then since we are dealing with local charts, we might assume that both \mathbf{t}_G and \mathbf{s}_G of $G_1 \Rightarrow G_0$ are just projections from \mathbb{R}^{n_1} to \mathbb{R}^{n_0} . A section of \mathbf{s}_G is a vector valued function $\mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_1}/\mathbb{R}^{n_0}$, and its being a bisection, namely also a section of \mathbf{t}_G , is an open condition. That is, we can always perturb a section to get a bisection. Let $U_{j'ij} := \mathbf{s}_G(\text{Im}(\bar{U}_{jij'} \rightarrow \bar{U}_{ij'})) \subset U_{j'}$ where $\bar{U}_{jij'}$ and $\bar{U}_{ij'}$ are defined as before in Lemma 3.7. If we can extend $\bar{g}_{ij}^{-1}\bar{g}_{ij'}$ and $g_{jj'}$ from $U_{j'ij} \cup e_{j'}(V_j \cap V_{j'})$ to a bisection $\bar{g}_{jj'}$ such that $\{\mathbf{s}_G(\bar{g}_{jj'})\}$ is an open set in $U_{j'}$, then we obtain a bisection $\bar{g}_{jj'}$ from $U_{jj'} := \bar{g}_{jj'}^{-1}(\{\mathbf{t}_G(\bar{g}_{jj'})\}) \cap U_j$ to $U_{j'j} := \{\mathbf{t}_G(\bar{g}_{jj'})\} \cap U_j$. It is easy to see that the $U_{jj'} \cong U_{j'j}$ are open in G_0 since $\{\mathbf{t}_G(\bar{g}_{jj'})\} \cong \{\mathbf{s}_G(\bar{g}_{jj'})\}$.

Therefore we are done as long as we can extend a smooth function f from the union of an open submanifold O with a closed submanifold V of an open set $B \subset \mathbb{R}^{n_0}$ to the whole B . Since V is closed, using its tubular neighborhood and partition of unity, we can first extend f from V to B as \tilde{f} . Then $f_1 = f - \tilde{f}|_{O \cup V}$ is 0 on V . We shrink the open set O a little bit to O_i such that $V \cap O \subset O_2 \subset O_1 \subset O$. Then we always have a smooth function p on B with $p|_{O_2} = 1$ and $p|_{B-O_1} = 0$. Then the extension function \tilde{f}_1 is defined by

$$\tilde{f}_1(x) = \begin{cases} f_1(x) \cdot p(x) & x \in O, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that \tilde{f}_1 is smooth, and it agrees with f_1 on O_2 and V because $V - O_2 = V - O_1 \subset B - O_1$ and $p|_{V-O_2} = 0$. Hence $\tilde{f} + \tilde{f}_1$ extends $f|_{O_2 \cup V}$. Now we extend the $\bar{g}_{ij}^{-1}\bar{g}_{ij'}$'s to $\bar{g}_{jj'}$'s; then the \bar{g} 's satisfy the cocycle condition on smaller open sets of the triple intersections $U_{j'ij}$ by construction.

Then we view $V_i \cup (\bigcup_{j: V_i \cap V_j \neq \emptyset} V_j)$ as one chart. Notice that a connected manifold is path connected. Also notice that we didn't use any topological property of V_i or U_i . This construction will eventually extend to the whole manifold M and obtain the desired \bar{g}_{ij} 's. Therefore we are again in the situation of a germ of manifolds and we can apply the proof of Lemma 3.7 to get the result. \square

3.2 The inverse map

In this section, we prove that the axioms involving the inverse map in the definition of stacky groupoid can be described by the multiplication and the identity.

Let $\mathcal{G} \Rightarrow M$ be a stacky groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$, and $G := G_1 \xrightarrow[\mathbf{t}_G]{\mathbf{s}_G} G_0$ a good groupoid presentation of \mathcal{G} as described in Section 3.1. So there is a map $e : M \rightarrow G_0$ presenting \bar{e} . We look at the diagram

$$\mathcal{G} \times_M \mathcal{G} \xrightarrow{m} \mathcal{G} \xleftarrow{\bar{e}} M$$

and its corresponding groupoid picture,

$$\begin{array}{ccccc} G_1 \times_{\mathbf{s}_G \mathbf{s}, M, \mathbf{t}_G \mathbf{t}} G_1 & & G_1 & & M \\ \parallel & & \parallel & & \parallel \\ & \swarrow J_l & E_m & \searrow J_r & \\ & G_0 \times_{\mathbf{s}, M, \mathbf{t}} G_0 & & G_0 & M \\ & & & \swarrow J_l & \searrow J_r \\ & & & E_{\bar{e}} & \end{array} \quad (13)$$

where E_m and $E_{\bar{e}} = G_1 \times_{\mathbf{t}_G, G_0, e} M$ are bibundles presenting the multiplication m and identity \bar{e} of \mathcal{G} respectively. We can form a left $G \times_{\mathbf{s}, M, \mathbf{t}} G$ module $E_m \times_{G_0} E_{\bar{e}}/G$. Examining the G action on $E_{\bar{e}}$, we see that the geometric quotient,

$$(E_m \times_{G_0} E_{\bar{e}})/G = E_m \times_{J_r, G_0, e} M, \quad \begin{array}{c} 0 \\ \nearrow \\ 1 \text{---} 2 \end{array} \quad (14)$$

is representable in \mathcal{C} by Lemma 3.11; and we see that the natural map $G_0 \times_{\mathbf{s}, M, \mathbf{s}} G_0 \xrightarrow{pr_2} G_0$ is a projection. This space should be pictured as the diagram above from the viewpoint of 2-groupoids. Moreover there is a left $G_1 \times_{\mathbf{s}_G \mathbf{s}, M, \mathbf{t}_G \mathbf{t}} G_1$ action (which might not be free and proper). Therefore, we can view it as a left G module with the left action of the first copy of G and a right G^{op} module with the left action of the second copy of G . Here G^{op} is G with the opposite groupoid structure.

Lemma 3.11. *The morphism $(pr_2 \circ J_l) \times J_r : E_m \rightarrow G_0 \times_{\mathbf{s}, M, \mathbf{s}} G_0$ is a projection.*

Proof. Let $f_1 : \mathcal{G} \times_{\bar{\mathbf{s}}, M, \bar{\mathbf{t}}} \mathcal{G} \rightarrow \mathcal{G} \times_{\bar{\mathbf{s}}, M, \bar{\mathbf{s}}} \mathcal{G}$ be given by $(g_1, g_2) \mapsto (g_1 \cdot g_2, g_2)$, i.e. $f_1 = m \times pr_2$; let $f_2 : \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G} \times_M \mathcal{G}$ be given by $(g_1, g_2) \mapsto (g_1 \cdot g_2^{-1}, g_2)$. Since we have

$$(g_1, g_2) \xrightarrow{f_1} (g_1 g_2, g_2) \xrightarrow{f_2} ((g_1 g_2) g_2^{-1}, g_2) \sim (g_1, g_2),$$

and

$$(g_1, g_2) \xrightarrow{f_2} (g_1 g_2^{-1}, g_2) \xrightarrow{f_1} ((g_1 g_2^{-1}) g_2, g_2) \sim (g_1, g_2),$$

$f_1 \circ f_2$ and $f_2 \circ f_1$ are isomorphic to id via 2-morphisms. Therefore f_1 is an isomorphism of stacks. Therefore $E_m \times_{pr_2 \circ J_l, G_0, \mathbf{t}_G} G_1$, presenting f_1 , is a Morita bibundle from the Lie groupoid $G_1 \times_M G_1 \rightrightarrows G_0 \times_M G_0$ to $G_1 \times_M G_1 \rightrightarrows G_0 \times_M G_0$. Hence the two moment maps J_l (of E_m) and $J_r \times \mathbf{s}_G$ are surjective projections. Moreover $J_r \times \mathbf{s}_G$ is invariant under the left groupoid action of $G_1 \times_M G_1$, so in particular under the action of the second copy. Notice that a G invariant projection $X \rightarrow Y$ descends to a projection $X/G \rightarrow Y$ if the G action is principal, in both of our two cases. As a result, the morphism $(pr_2 \circ J_l) \times J_r : E_m \rightarrow G_0 \times_M G_0$ is a projection.

Moreover since the left groupoid action of $G_1 \times_M G_1$ is principal on the bibundle $E_m \times_{pr_2 \circ J_l, G_0, \mathbf{t}_G} G_1$, the induced action of the first copy of G_1 on the quotient E_m is principal. \square

Lemma 3.12. *The bibundle (14) is a Morita equivalence from G to G^{op} with moment maps $pr_1 \circ J_l$ and J_r .*

Proof. The left action of G is principal followed by the principal action of G on E_m proven in Lemma 3.11, and the proof of the principality of the G^{op} action is similar (one considers $G_1 \times_{G_0} E_m$). \square

Remark 3.13. Another fibre product $E_m \times_{pr_2 \circ J_l, G_0, e} M$ is isomorphic to G_1 trivially via b_r . But the morphisms we use to construct the fibre product are different.

Notice that using the inverse operation, a G^{op} module is also a G module. In other words, the above lemma says that $E_m \times_{J_r, G_0, e} M$ is a Morita bibundle between G and G where the right G action is via the left action of the second copy of $G \times_M G$ composed with the inverse. With this viewpoint, we have a stronger statement:

Lemma 3.14. *As Morita bibundles from G to G , $E_m \times_{J_r, G_0, e} M$ and E_i are isomorphic.*

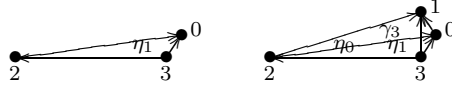
Proof. We know from the property of E_i that $g \cdot g^{-1} \sim 1$; that is, there is an isomorphism of H.S. bibundles

$$((G_1 \times_{s_G, G_0, J_l} E_i) \times_{t_G \times J_r, G_0 \times_M G_0} E_m) / G_1 \times_M G_1 \cong G_0 \times_{eot, G_0, t_G} G_1,$$

where $G_0 \times_{eot, G_0, t_G} G_1$ presents the map $e \circ \bar{t} : \mathcal{G} \rightarrow M \rightarrow \mathcal{G}$. We will first show that $E_m \times_{G_0} M$ also has this property.

Let $(\gamma_3, \eta_1, \eta_0) \in ((G_1 \times_{G_0} (E_m \times_{J_r, G_0, e} M)) \times_{G_0 \times_M G_0} E_m)$ (see (15)).

(15)



Since the right action of G_1 on E_m is principal (now viewing E_m as a bibundle from $G \times_M G$ to G), we have an isomorphism

$$\Phi : E_m \times_{J_l, G_0 \times_M G_0, J_l} E_m \cong E_m \times_{J_r, G_0, t_G} G_1. \quad (16)$$

The right $G_1 \times_M G_1$ action is

$$(\gamma_3, \eta_1, \eta_0) \cdot (\gamma_1, \gamma_2) = (\gamma_3 \cdot \gamma_1, (1, \gamma_2^{-1}) \cdot \eta_1, (\gamma_1, \gamma_2)^{-1} \cdot \eta_0). \quad (17)$$

Noticing that

$$J_l(\eta_1) = J_l((\gamma_3, 1)\eta_0) = (s_G(\gamma_3), pr_2(J_l(\eta_0)) = pr_2(J_l(\eta_1))),$$

we have a morphism in \mathcal{C} ,

$$\tilde{\phi} : (G_1 \times_{G_0} (E_m \times_{J_r, G_0, e} M)) \times_{G_0 \times_M G_0} E_m \rightarrow G_0 \times_{eot, G_0, t_G} G_1,$$

by

$$(\gamma_3, \eta_1, \eta_0) \mapsto (t_G(\gamma_3), pr_G \circ \Phi(\eta_1, (\gamma_3, 1)\eta_0)).$$

Further, $\tilde{\phi}$ is invariant under the right action (17) because the right action and left action on a bibundle commute. Therefore, $\tilde{\phi}$ descends to a morphism in \mathcal{C} ,

$$\phi : ((G_1 \times_{G_0} (E_m \times_{J_r, G_0, e} M)) \times_{G_0 \times_M G_0} E_m) / G_1 \times_M G_1 \rightarrow G_0 \times_{\text{eot}, G_0, \mathbf{t}_G} G_1.$$

Moreover, ϕ is an isomorphism by (16) and the fact that the first copy G_1 acts on G_1 by multiplication. It is not hard to check that ϕ is equivariant and commutes with the moment maps of the bibundles. Therefore,

$$((G_1 \times_{G_0} (E_m \times_{J_r, G_0, e} M)) \times_{G_0 \times_M G_0} E_m) / G_1 \times_M G_1 \cong G_0 \times_{\text{eot}, G_0, \mathbf{t}_G} G_1$$

as H.S. bibundles. One proceeds similarly to prove the other symmetric isomorphism corresponding to $g^{-1} \cdot g \sim 1$.

Let φ be the composed isomorphism

$$((G_1 \times_{G_0} (E_m \times_{J_r, G_0, e} M)) \times_{G_0 \times_M G_0} E_m) / G_1 \times_M G_1 \rightarrow ((G_1 \times_{G_0} E_i) \times_{G_0 \times_M G_0} E_m) / G_1 \times_M G_1. \quad (18)$$

Suppose $\varphi([(1_g, \eta_1, \eta_2)]) = [(1_g, \tilde{\eta}_1, \tilde{\eta}_2)]$ (we can still assume that the first component is 1 because the $G_1 \times_M G_1$ action on both sides is right multiplication by the first copy; we can assume that they are 1 at the same point because φ commutes with the moment maps on the left leg). Examining the morphisms inside the fibre products, we have

$$pr_1 \circ J_l(\eta_2) = \mathbf{t}_G(1_g) = pr_1 \circ J_l(\tilde{\eta}_2) = g.$$

Since φ commutes with the moment maps on the right leg, we have

$$J_r(\eta_2) = J_r(\tilde{\eta}_2).$$

Similarly to the proof of Lemma 3.11, we can show that $G_1 \times_{\mathbf{s}_G, G_0, pr_1 \circ J_l} E_m$ is a Morita bibundle from $G \times_M G$ to $G \times_M G$. Then $(1_g, \eta_2)$ and $(1_g, \tilde{\eta}_2)$ are both in $G_1 \times_{\mathbf{s}_G, G_0, pr_1 \circ J_l} E_m$ and their images under the right moment map $\mathbf{s}_G \times J_r$ are both $(g, J_r(\eta_2))$. By principality of this left $G_1 \times_M G_1$ action, there is a unique $(\gamma_1, \gamma_2) \in G_1 \times_M G_1$ such that

$$(\gamma_1, \gamma_2) \cdot (1, \eta_2) = (1, \tilde{\eta}_2).$$

Therefore $\gamma_1 = 1$ and $(1, \gamma_2) \cdot \eta_2 = \tilde{\eta}_2$. This left $G_1 \times_M G_1$ action on E_m is exactly the left $G_1 \times_M G_1$ action on the second copy of E_m in (18). Using this γ_2 , we have

$$(1, \tilde{\eta}_1, \tilde{\eta}_2) \cdot (1, \gamma_2) = (1, \gamma_2^{-1}) \cdot (1, \tilde{\eta}_1, \tilde{\eta}_2) = (1, \eta'_1, \eta_2).$$

Therefore the isomorphism

$$\varphi : [(1_g, \eta_1, \eta_2)] \mapsto [(1_g, \eta'_1, \eta_2)]$$

induces a map $\psi : E_m \times_{G_0} M \rightarrow E_i$ by $\eta_1 \mapsto \eta'_1$. It's routine to check that ψ is an isomorphism of Morita bibundles. \square

We have seen in this lemma that the 2-identities satisfied by E_i are actually naturally satisfied by $E_m \times_{J_r, G_0, e} M$. Notice that for the first part of the proof, we didn't use any information involving the inverse map. Our conclusion is that the inverse map represented by E_i can be replaced by $E_m \times_{J_r, G_0, e} M$ without any further conditions (not even on the 2-morphisms) because the natural 2-morphisms coming along with the bibundle $E_m \times_{J_r, G_0, e} M$ naturally go well with the other 2-morphisms, the a 's and b 's.

Proposition 3.15. *A stacky groupoid \mathcal{G} in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ can also be defined by replacing the axioms involving inverses by the axiom that*

$E_m \times_{J_r, G_0, e} M$ is a Morita bibundle from G to G^{op} for some good presentation G of \mathcal{G} .

Proof. It is clear from Lemma 3.14 that the existence of the inverse map guarantees that the bibundle $E_m \times_{J_r, G_0, e} M$ is a Morita bibundle from G to G^{op} for a good presentation G of \mathcal{G} .

On the other hand, if $E_m \times_{J_r, G_0, e} M$ is a Morita bibundle from G to G^{op} for some presentation G of \mathcal{G} , then we construct the inverse map $i : \mathcal{G} \rightarrow \mathcal{G}$ by this bibundle. Because of the nice properties of $E_m \times_{J_r, G_0, e} M$ that we have proven in the first half of Lemma 3.14, this newly defined inverse map satisfies all the axioms that the inverse map satisfies. \square

Remark 3.16. This theorem holds also for W-groupoids and the proof is similar.

Remark 3.17. There is similar treatment of the antipode in hopfish algebras [35]. In fact SLie groups are a geometric version of hopfish algebras. The geometric quotient (14) corresponds to $\text{hom}_{\mathcal{A}}(\epsilon, \Delta)$ in the case of hopfish algebra. Thus the new definition of SLie group modulo 2-morphisms is analogous to the definition of hopfish algebra.

Sometimes the inverse map of a stacky groupoid is given by a specific groupoid isomorphism $i : G \rightarrow G$ on some presentation (for example $\mathcal{G}(A)$ and $\mathcal{H}(A)$ in [36] and (quasi-)Hopf algebras as the algebra counter-part).

Lemma 3.18. *The inverse map of a stacky groupoid \mathcal{G} in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ is given by a groupoid isomorphism $i : G \rightarrow G$ for some presentation G if and only if on this presentation, $E_m \times_{J_r, G_0, e} M$ is a trivial right G principal bundle over G_0 .*

Proof. It follows from Lemma 3.14 and the fact that the inverse is given by a morphism $i : G \rightarrow G$ if and only if the bibundle E_i is trivial. \square

4 2-groupoids and stacky groupoids

4.1 From stacky groupoids to 2-groupoids

Suppose $\mathcal{G} \Rightarrow M$ is a stacky groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$; in this section we construct a corresponding 2-groupoid object $X_2 \Rrightarrow X_1 \Rrightarrow X_0$ in $(\mathcal{C}, \mathcal{T}'')$. When $\mathcal{G} \Rightarrow M$ is an SLie groupoid, what we construct is a Lie 2-groupoid. When $\mathcal{G} \Rightarrow M$ is further a W-groupoid, the corresponding Lie 2-groupoid is 2-étale; that is, the maps $X_2 \rightarrow \text{hom}(\Lambda[2, j], X)$ are étale for $j = 0, 1, 2$.

Theorem 4.1. *A stacky groupoid object $\mathcal{G} \Rightarrow M$ in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ with a good chart G_0 of \mathcal{G} corresponds to a 2-groupoid object $X_2 \Rrightarrow X_1 \Rrightarrow X_0$ in $(\mathcal{C}, \mathcal{T}'')$.*

A W-groupoid with a good étale chart corresponds to a 2-étale Lie 2-groupoid.

The construction of $X_2 \Rightarrow X_1 \Rightarrow X_0$ Given a stacky groupoid object $\mathcal{G} \Rightarrow M$ in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ and a good groupoid presentation $G_1 \Rightarrow G_0$ of \mathcal{G} , let E_m be the H.S. bimodule presenting the morphism m . Let $J_l : E_m \rightarrow G_0 \times_M G_0$ and $J_r : E_m \rightarrow G_0$ be the moment maps of the bimodule E_m . Notice that for a stacky groupoid, $g \cdot 1 \simeq g$ up to a 2-morphism; that is, $m|_{\mathcal{G} \times_M M} \simeq id$ up to a 2-morphism. Translating this into groupoid language, $J_l^{-1}(G_0 \times_M M)$ and G_1 are the H.S. bimodules presenting $m|_{\mathcal{G} \times_M M}$ and id respectively. By the definition of stacky groupoids, the isomorphism is provided by $b_r : J_l^{-1}(G_0 \times_M M) \rightarrow G_1$. Similarly, we have the isomorphism $b_l : J_l^{-1}(M \times_M G_0) \rightarrow G_1$.

We construct

$$X_0 = M, X_1 = G_0, X_2 = E_m$$

with the structure maps

$$\begin{aligned} d_0^1 = s, d_1^1 = t : X_1 \rightarrow X_0, \quad d_0^2 = pr_2 \circ J_l, d_1^2 = J_r, d_2^2 = pr_1 \circ J_l : X_2 \rightarrow X_1, \\ s_0^0 = e : X_0 \rightarrow X_1, \quad s_0^1 = b_l^{-1} \circ e_G, s_1^1 = b_r^{-1} \circ e_G : X_1 \rightarrow X_2 \end{aligned} \quad (19)$$

where pr_i is the i -th projection $G_0 \times_M G_0 \rightarrow G_0$. Item 4b in Def. 3.4 implies that $s_0^1 \circ s_0^0 = s_1^1 \circ s_0^0$. The other coherence conditions in (1) are implied by the fact that the 2-morphism preserves moment maps. We still need the 3-multiplication maps

$$m_i : \Lambda(X)_{3,i} \rightarrow X_2 \quad i = 0, \dots, 3.$$

Let us first construct m_0 . Notice that in the 2-associative diagram, we have a 2-morphism $a : m \circ (m \times id) \rightarrow m \circ (id \times m)$. Translating this into the language of groupoids, we have the following isomorphism of bimodules:

$$a : ((E_m \times_{G_0} G_1) \times_{G_0 \times_M G_0} E_m) / (G_1 \times_M G_1) \rightarrow ((G_1 \times_{G_0} E_m) \times_{G_0 \times_M G_0} E_m) / (G_1 \times_M G_1). \quad (20)$$

The plan of proof is to take $(\eta_1, \eta_2, \eta_3) \in \Lambda(X)_{3,0}$. Then $(\eta_3, 1, \eta_1)$ represents a class in $(E_m \times_{G_0} G_1) \times_{G_0 \times_M G_0} E_m / \sim$ (we write \sim when it is clear which groupoid action is meant). Moreover, its image under a can be represented by $(1, \eta_0, \eta_2)$; that is,

$$a([(\eta_3, 1, \eta_1)]) = [(1, \eta_0, \eta_2)].$$

Then we arrive naturally at η_0 .

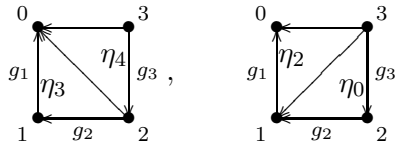
Now we prove it strictly. To simplify our notation, we call the left and right hand sides of (20) L and R respectively. Since the action on G_1 's is by multiplication, we have G_1 principal bundles $\tilde{L} \rightarrow L = \tilde{L}/G_1$ and $\tilde{R} \rightarrow R = \tilde{R}/G_1$, where

$$\tilde{L} = E_m \times_{J_r, G_0, pr_1 J_l} E_m, \quad \tilde{R} = E_m \times_{J_r, G_0, pr_2 J_l} E_m,$$

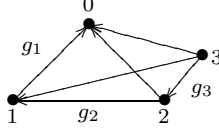
with G_1 principal actions

$$(\eta_3, \eta_1) \cdot \gamma' = (\eta_3 \gamma', (\gamma', 1)^{-1} \eta_1), \quad (\eta_0, \eta_2) \cdot \gamma' = (\eta_0 \gamma', (1, \gamma')^{-1} \eta_2);$$

they are presented by diagrams



which all together fit inside



We imagine that the j -dimensional faces of the picture are elements of X_j . We also put g_i 's in the picture to help. We view $a : (g_1 g_2) g_3 \rightarrow g_1 (g_2 g_3)$, and $\eta_3 \in E_m$ is responsible for $g_1 g_2$, etc.

Similarly to Lemma 3.11, $(pr_1 \circ J_l) \times J_r : E_m \rightarrow G_0 \times_{\mathbf{t}, M, \mathbf{t}} G_0$ is a G principal bundle with left G action induced by the second copy of the $G \times_M G$ bibundle action on E_m . Hence we have

$$E_m \times_{G_0 \times_M G_0} E_m \cong G_1 \times_{\mathbf{s}_G, G_0, pr_2 \circ J_l} E_m, \quad (\tilde{\eta}_2, \eta_2) \mapsto (\gamma, \eta_2), \quad \text{with } \tilde{\eta}_2 = (1, \gamma) \eta_2$$

which gives rise to an isomorphism $\tilde{\phi}$ in \mathcal{C} ,

$$E_m \times_{J_r, G_0, pr_2 J_l} E_m \times_{G_0 \times_M G_0} E_m \cong E_m \times_{J_r, G_0, pr_2 J_l} E_m \times_{pr_2 J_l, G_0, \mathbf{s}_G} G_1,$$

given by

$$\tilde{\phi}(\eta_0, \tilde{\eta}_2, \eta_2) = (\eta_0 \gamma, \eta_2, \gamma).$$

Moreover $\tilde{\phi}$ is G equivariant w.r.t. the following G actions

$$(\eta_0, \tilde{\eta}_2, \eta_2) \cdot \gamma' = (\eta_0 \gamma', (1, \gamma')^{-1} \tilde{\eta}_2, \eta_2), \quad (\eta_0, \eta_2, \gamma) \cdot \gamma' = (\eta_0, \eta_2, \gamma'^{-1} \gamma).$$

Hence it gives an isomorphism in \mathcal{C} between the quotients,

$$\phi : R \times_{G_0 \times_M G_0, J_l} E_m \cong \tilde{R}.$$

We have a commutative diagram

$$\begin{array}{ccc} \Lambda(X)_{3,0} & \xrightarrow{(\eta_1, \eta_2, \eta_3) \mapsto \eta_2} & E_m \\ \downarrow (\eta_1, \eta_2, \eta_3) \mapsto [(\eta_1, \eta_3)] & & \downarrow J_l \\ L & \xrightarrow{a} R \longrightarrow G_0 \times_M G_0. \end{array} \quad (21)$$

Hence there exists a morphism in \mathcal{C} from $\Lambda(X)_{3,0}$ to the fibre product $R \times_{G_0 \times_M G_0, J_l} E_m \cong \tilde{R}$, and m_0 is defined as the composition of morphisms $\Lambda(X)_{3,0} \rightarrow \tilde{R} \xrightarrow{pr_1} E_m$.

For other m 's, we precede in a similar fashion. More precisely, for m_1 one can make the same definition as for m_0 but using a^{-1} . It is even easier to define m_2 and m_3 . Thus we realize that given any three η 's, we can always put them in the same spots as we did for m_0 . Then any three of them determine the fourth. Hence the m 's are compatible with each other.

Proof that what we construct is a 2-groupoid By Prop-Def. 2.16, to show that the above construction gives us a 2-groupoid in $(\mathcal{C}, \mathcal{T}'')$, we just have to show that the m_i 's satisfy the coherence conditions, associativity and the 1-Kan and 2-Kan conditions. Condition 1-Kan is implied by the fact that $\mathbf{s}, \mathbf{t} : G_0 \rightrightarrows M$ are projections; $Kan(2, 1)$ is implied by the fact that the moment map $J_l : E_m \rightarrow G_0 \times_{\mathbf{s}, M, \mathbf{t}} G_0$ is a projection; $Kan(2, 2)$ is implied by Lemma 3.11; and $Kan(2, 0)$ can be proven similarly.

The coherence conditions The first identity in eq. (9) corresponds to an identity of 2-morphisms,

$$(1 \cdot (g_2 \cdot g_3) \stackrel{a}{\sim} (1 \cdot g_2) \cdot g_3 \sim g_2 \cdot g_3) = (1 \cdot (g_2 \cdot g_3) \sim g_1 \cdot g_2).$$

More precisely, restrict the two bimodules in (20) to $M \times_M G_0 \times_M G_0$; then we get E_m on the left hand side because $J_l^{-1}(M \times_M G_0) \stackrel{b_l}{\cong} G_1$ and $((G_1 \times_M G_1) \times_{G_0 \times_M G_0} E_m)/G_1 \times_M G_1 = E_m$. In fact, the elements in $(E_m \times_{G_0} G_1) \times_{G_0 \times_M G_0} E_m|_{M \times_M G_0 \times_M G_0}/\sim$ have the form $[(s_0 \circ d_2(\eta), 1, \eta)]$, and the isomorphism to E_m is given by $[(s_0 \circ d_2(\eta), 1, \eta)] \mapsto \eta$. Similarly for the right hand side; i.e., $[(s_0 \circ d_1(\eta), 1, \eta)] \mapsto \eta$ gives the other isomorphism. By 4d in Def. 3.4, the composition of the first and the inverse of the second map is a (restricted to the restricted bimodules), so we have

$$a([(s_0 \circ d_2(\eta), 1, \eta)]) = ([(1, \eta, s_0 \circ d_1(\eta))]),$$

which implies the first identity in (9). The rest follows similarly.

Associativity To prove the associativity, we use the cube condition 3c in Def. 3.4. Let η_{ijk} 's denote the faces in X_2 fitting in diagram (22). Suppose we are given the faces $\eta_{0i4} \in X_2$ and the faces $\eta_{0ij} \in X_2$. Then we have two ways to determine the face η_{123} using m 's as described in Prop-Def. 2.16. We will show below that these two constructions give the same element in X_2 .

Translate the cube condition into the language of groupoids. The morphisms become H.S. bibundles and the 2-morphisms become the morphisms between these bibundles. The cube condition tells us that the following two compositions of morphisms are the same (where for simplicity, we omit the base space of the fibre products and the groupoids by which we take quotients):

$$\begin{aligned} (E_m \times G_1 \times G_1) \times (E_m \times G_1) \times E_m / \sim & \longleftrightarrow ((g_1 g_2) g_3) g_4 \\ \xrightarrow{id \times a} (E_m \times G_1 \times G_1) \times (G_1 \times E_m) \times E_m / \sim & \longleftrightarrow (g_1 g_2) (g_3 g_4) \\ \xrightarrow{id} (G_1 \times G_1 \times E_m) \times (E_m \times G_1) \times E_m / \sim & \longleftrightarrow (g_1 g_2) (g_3 g_4) \\ \xrightarrow{id \times a} (G_1 \times G_1 \times E_m) \times (G_1 \times E_m) \times E_m / \sim & \longleftrightarrow g_1 (g_2 (g_3 g_4)) \end{aligned}$$

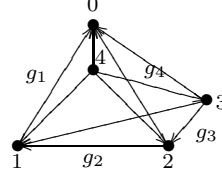
and

$$\begin{aligned} (E_m \times G_1 \times G_1) \times (E_m \times G_1) \times E_m / \sim & \longleftrightarrow ((g_1 g_2) g_3) g_4 \\ \xrightarrow{a \times id} (G_1 \times E_m \times G_1) \times (E_m \times G_1) \times E_m / \sim & \longleftrightarrow (g_1 (g_2 g_3)) g_4 \\ \xrightarrow{id \times a} (G_1 \times E_m \times G_1) \times (G_1 \times E_m) \times E_m / \sim & \longleftrightarrow g_1 ((g_2 g_3) g_4) \\ \xrightarrow{a \times id} (G_1 \times G_1 \times E_m) \times (G_1 \times E_m) \times E_m / \sim & \longleftrightarrow g_1 (g_2 (g_3 g_4)). \end{aligned}$$

Tracing the element $(\eta_{034}, (\eta_{023}, 1), (\eta_{012}, 1, 1))$ through the first and second composition, it

should end up as the same element. So we have

$$\begin{aligned}
& [((\eta_{012}, 1, 1)), (\eta_{023}, 1), \eta_{034}] \\
& \xrightarrow{id \times a} [((\eta_{012}, 1, 1), (1, \eta_{234}), \eta_{024})] \\
& \xrightarrow{id} [((1, 1, \eta_{234}), (\eta_{012}, 1), \eta_{024})] \\
& \xrightarrow{id \times a} [((1, 1, \eta_{234}), (1, \eta_{124}), \eta_{014})],
\end{aligned}
\tag{22}$$



where by definition of m_0 , $\eta_{234} = m_0(\eta_{034}, \eta_{024}, \eta_{023})$ and $\eta_{124} = m_0(\eta_{024}, \eta_{014}, \eta_{012})$, and

$$\begin{aligned}
& [((\eta_{012}, 1, 1), (\eta_{023}, 1), \eta_{034})] \\
& \xrightarrow{a \times id} [((1, \eta_{123}, 1), (\eta_{013}, 1), \eta_{034})] \\
& \xrightarrow{id \times a} [((1, \eta_{123}, 1), (1, \eta_{134}), \eta_{014})] \\
& \xrightarrow{a \times id} [((1, 1, \eta_{234}), (1, \eta_{124}), \eta_{014})],
\end{aligned}$$

where by definition of m_0 , $\eta_{123} = m_0(\eta_{023}, \eta_{013}, \eta_{012})$ and $\eta_{134} = m_0(\eta_{034}, \eta_{014}, \eta_{013})$. Therefore, the last map tells us that

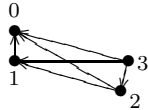
$$\eta_{123} = m_3(\eta_{234}, \eta_{134}, \eta_{124}).$$

Therefore associativity holds!

Comments on the étale condition It is easy to see that if $G_1 \Rightarrow G_0$ is an étale Lie groupoid, by principality of the right G action on E_m , the moment map $E_m \rightarrow G_0 \times_M G_0$ is an étale Lie groupoid. Moreover since $E_m \rightarrow \Lambda(X)_{2,j} = \Lambda[2, j](X)$ is a surjective submersion by $Kan(2, j)$, by dimension counting, it is furthermore an étale map.

4.2 From 2-groupoids to stacky groupoids

If X is a 2-groupoid object in $(\mathcal{C}, \mathcal{T}'')$, then $G_1 := d_2^{-1}(s_0(X_0)) \subset X_2$, which is the set of bigons, is a groupoid over $G_0 := X_1$ (Lemma 4.3). Here we might notice that there is another natural choice for the space of bigons, namely $\tilde{G}_1 := d_0^{-1}(s_0(X_0))$. But $G_1 \cong \tilde{G}_1$ by the following observation: given an element $\eta_3 \in G_1$, it fits in the following picture,



In this picture, $1 \rightarrow 0$ and $2 \rightarrow 3$ are degenerate, and η_2, η_1 are degenerate.

Then m_0 gives a morphism

$$\varphi : G_1 \rightarrow \tilde{G}_1, \tag{23}$$

and m_3 gives the inverse. Therefore we might consider only G_1 . Then $G_1 \Rightarrow G_0$ presents a stack which has an additional groupoid structure.

Theorem 4.2. *A 2-groupoid object X in $(\mathcal{C}, \mathcal{T}'')$ corresponds to a stacky groupoid object $\mathcal{G} \Rightarrow X_0$ with a good chart in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$, where \mathcal{G} is presented by the groupoid object $G_1 \Rightarrow G_0$.*

A 2-étale Lie 2-groupoid corresponds to a W -groupoid with a good étale chart.

We prove this theorem by several lemmas.

About the stack \mathcal{G}

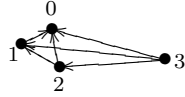
Lemma 4.3. $G_1 \Rightarrow G_0$ is a groupoid object in $(\mathcal{C}, \mathcal{T}'')$.

Proof. The target and source maps are given by d_0^2 and d_1^2 . The identity $G_0 \rightarrow G_1$ is given by $s_0^1 : X_1 \rightarrow X_2$. The image of s_0^1 is in $G_1 (\subset X_2)$. Their compatibility conditions are implied by the compatibility conditions of the structure maps of simplicial manifolds. Since G_1 is the pull-back of $X_2 \xrightarrow{d_1 \times d_2} X_1 \times_{d_1, X_0, d_0} X_1$ by the map

$$X_1 \rightarrow X_1 \times_{d_1, X_0, d_0} X_1, \quad \text{with } g \mapsto (s_0(d_1(g)), g),$$

$s_G = d_1 : G_1 \rightarrow X_1$ is a surjective projection. Similarly t_G is also a surjective projection.

The multiplication is given by the 3-multiplication of X .



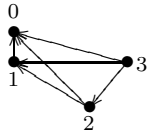
In this picture, $\eta_3 = s_0 \circ s_0(d_0^1 \circ d_2^2(\eta_2))$ is the degenerate face corresponding to the point 0 ($= 1 = 2$).

More precisely, any $(\eta_0, \eta_2) \in G_1 \times_{s_G, G_0, t_G} G_1$ fits in the above picture. We define $\eta_0 \cdot \eta_2 = m_1(\eta_0, \eta_2, \eta_3)$. Then the associativity of the 3-multiplications ensures the associativity of “ \cdot ”. The inverse is also given by 3-multiplications: $\eta_2^{-1} = m_0(\eta_1, \eta_2, \eta_3)$ with $\eta_1 = s_0^1(d_1^2(\eta_2))$ the degenerate face in $s_0^1(X_1)$. It is clear from the construction that all the structure maps are morphisms in \mathcal{C} . \square

Remark 4.4. A similar construction shows that $\tilde{G}_1 \Rightarrow G_0$, with $t = d_2^1$ and $s = d_1^1$, is a groupoid object in $(\mathcal{C}, \mathcal{T}'')$ isomorphic to $G_1 \Rightarrow G_0$ via the map φ^{-1} (see equation (23)).

Proof that $\mathcal{G} \Rightarrow M$ is a stacky groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$

Source and target maps There are three maps $d_i^2 : X_2 \rightarrow X_1 = G_0$ and they (as the moment maps of the action) each correspond to a groupoid action. The actions are similarly given by the 3-multiplications as the multiplication of G_1 . The axioms of the actions are given by the associativity. For example, for d_1^2 , any $(\eta_0, \eta_2) \in X_2 \times_{d_1^2, X_1, t_G} G_1$ fits inside the following picture:



In this picture, $1 \rightarrow 0$ is a degenerate edge and $\eta_3 = s_0 \circ d_2^2(\eta_2)$ is a degenerate face.

(24)

Then

$$\eta_0 \cdot \eta_2 := m_1(\eta_0, \eta_2, s_0 d_2^2(\eta_2)). \quad (25)$$

Moreover, notice that the four ways to compose source, target and face maps $G_1 \xrightarrow{s_G} G_0 \xrightarrow{d_0^1} X_0$ and $G_1 \xrightarrow{t_G} G_0 \xrightarrow{d_1^1} X_0$ only give two different maps: $d_0^1 s_G$ and $d_1^1 t_G$. They are surjective projections since the d_i^1 's, s_G and t_G are such, and they give the source and target maps $\bar{s}, \bar{t} : \mathcal{G} \Rightarrow X_0$ where \mathcal{G} is the presentable stack presented by $G_1 \Rightarrow G_0$. Therefore \bar{s} and \bar{t} are also surjective projections (similarly to Lemma 4.2 in [36]). We use these two maps to form the product groupoid

$$G_1 \times_{d_0^1 s_G, X_0, d_1^1 t_G} G_1 \Rightarrow G_0 \times_{d_0^1, X_0, d_1^1} G_0 \quad (26)$$

which presents the stack $\mathcal{G} \times_{\bar{s}, X_0, \bar{t}} \mathcal{G}$.

Multiplication

Lemma 4.5. $(X_2, d_2^2 \times d_0^2, d_1^2)$ is an H.S. bimodule from the product groupoid (26) to $G_1 \rightrightarrows G_0$.

Proof. By $Kan(2, 1)$, $d_2^2 \times d_1^2$ is a surjective projection from X_2 to $G_0 \times_{d_0^1, X_0, d_1^1} G_0$, so we only have to show that the right action of $G_1 \rightrightarrows G_0$ on X_2 is free and transitive. This is implied by $Kan(3, j)$ and $Kan(3, j)!$ respectively.

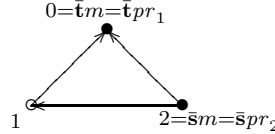
Transitivity: any (η_1, η_0) such that $d_0^2(\eta_1) = d_0^2(\eta_0)$ and $d_2^2(\eta_0) = d_2^2(\eta_1)$ fits inside picture (24). Then there exists $\eta_2 := m_2(\eta_0, \eta_1, \eta_3) \in G_1$, making $\eta_0 \cdot \eta_2 = \eta_1$.

Freeness: if $(\eta_0, \eta_2) \in X_2 \times_{d_1, X_1, t_G} G_1$ satisfies $\eta_0 \cdot \eta_2 (= m_1(\eta_0, \eta_2, \eta_3)) = \eta_0$, then $\eta_2 = m_2(\eta_0, \eta_0, \eta_3)$, and η_3 is degenerate. Thus by 2a in Prop-Def. 2.16, $m_2(\eta_0, \eta_0, \eta_3) = s_0^1(3 \rightarrow 1)$ is a degenerate face. Therefore $\eta_2 = 1$. \square

Therefore X_2 gives a morphism $m : \mathcal{G} \times_{X_0} \mathcal{G} \rightarrow \mathcal{G}$.

Lemma 4.6. With the source and target maps constructed above, m is a multiplication of $\mathcal{G} \rightrightarrows X_0$.

Proof. By construction, it is clear that $\bar{t} \circ m = \bar{t} \circ pr_1$ and $\bar{s} \circ m = \bar{s} \circ pr_2$, where $pr_i : \mathcal{G} \times_{\bar{s}, X_0, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}$ is the i -th projection (see the picture below).



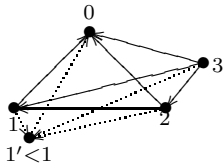
To show the associativity, we reverse the argument in Section 4.1. There, we used the 2-morphism a to construct the 3-multiplications. Now we use the 3-multiplications and their associativity to construct a . Given the two H.S. bibundles presenting $m \circ (m \times id)$ and $m \circ (id \times m)$ respectively, we want to construct a map a as in (20), where $E_m = X_2$ and $M = X_0$. Given any element in $(X_2 \times_{G_0} G_1) \times_{G_0 \times_{X_0} G_0} X_2 / G_1 \times_{X_0} G_1$, as in Section 4.1, the idea is that we can write it in the form of $[(\eta_3, 1, \eta_1)]$, with $(\eta_1, \eta_2, \eta_3) \in \text{hom}(\Lambda[3, 0], X)$ for some η_2 . Then we define

$$a([(\eta_3, 1, \eta_1)]) := [(1, \eta_0 := m_0(\eta_1, \eta_2, \eta_3), \eta_2)].$$

As before, we need to strictify the proof via diagram chasing. Similarly to (21), we have

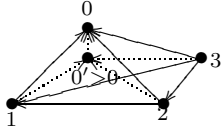
$$\begin{array}{ccccc} \text{hom}(\Lambda[3, 0], X) & \xrightarrow{(\eta_1, \eta_2, \eta_3) \mapsto (\eta_0, \eta_2)} & \tilde{R} & \longrightarrow & E_m \\ (\eta_1, \eta_2, \eta_3) \mapsto [(\eta_1, \eta_3)] \downarrow & & \downarrow & & \downarrow J_l \\ L & \xrightarrow{\quad a \quad} & R & \longrightarrow & G_0 \times_M G_0 \end{array} \quad (27)$$

Hence we should show that the definition of a does not depend on the choice of η_1, η_3 and η_2 set-theoretically, and then a is a morphism in \mathcal{C} . We first show the first statement. First of all (see the picture below),



In this picture, $1 \rightarrow 1'$ is a degenerate edge and $\eta_{01'1}, \eta_{1'12}$ are degenerate faces.

if we choose a different $\tilde{\eta}_2$, since (η_1, η_2, η_3) and $(\eta_1, \tilde{\eta}_2, \eta_3)$ are both in $\text{hom}(\Lambda[3, 0], X)$, we have $d_2^2(\eta_2) = d_2^2(\tilde{\eta}_2)$ and $d_1^2(\eta_2) = d_1^2(\tilde{\eta}_2)$. So $\eta_2 = \eta_{013}$ and $\tilde{\eta}_2 = \eta_{01'3}$ form a degenerate horn. By $Kan(3, 0)$ there exists $\gamma = \gamma_{1'13} \in G_1$ such that $(1, \gamma) \cdot \tilde{\eta}_2 (= \gamma \cdot \tilde{\eta}_2) = \eta_2$, that is $\eta_{013} = m_1(\gamma, \eta_{01'3}, s_1^1(0 \rightarrow 1))$. Then by associativity and the definition of the right G_1 action (25), we have $m_0(\eta_{023}, \eta_{01'3}, \eta_{01'2}) = \eta_{1'23} = \eta_{123} \cdot \gamma$. Therefore we have $[(1, \eta_{1'23}, \tilde{\eta}_2)] = [(1, \eta_{123}, \eta_2)]$. So the choice of η_2 will not affect the definition of a . Secondly (see the following picture),



In this picture, $0' \rightarrow 0$ is a degenerate edge and $\eta_{00'1}, \eta_{00'3}$ are degenerate faces.

if we choose a different $(\tilde{\eta}_3 = \eta_{0'12}, 1, \tilde{\eta}_1 = \eta_{0'23})$, such that $\eta_3 = \tilde{\eta}_3 \cdot \gamma_{00'2}$ and $\tilde{\eta}_1 = (\gamma_{00'2}, 1) \cdot \eta_1 = \gamma_{00'2} \cdot \eta_1$ for a $\gamma_{00'2} \in G_1$, then by associativity we have $(\tilde{\eta}_1, \eta_2, \tilde{\eta}_3) \in \text{hom}(\Lambda[3, 0], X)$ and

$$m_0(\tilde{\eta}_1, \eta_2, \tilde{\eta}_3) = \eta_{123} = m_0(\eta_1, \eta_2, \eta_3).$$

So this choice will not affect a either.

In all our cases, for a set-theoretical map to be a morphism in \mathcal{C} , we only have to verify it \mathcal{T} -locally. Luckily, our surjective projections do have \mathcal{T} -local sections and $\text{hom}(\Lambda[3, 0], X) \rightarrow L$, being a composition of two surjective projections $\text{hom}(\Lambda[3, 0], X) = \tilde{L} \times_{G_0 \times_M G_0} E_m \rightarrow \tilde{L}$ and $\tilde{L} \rightarrow L$, is a surjective projection.

Now the higher coherence of a follows from the associativity by the same argument as in Section 4.1. \square

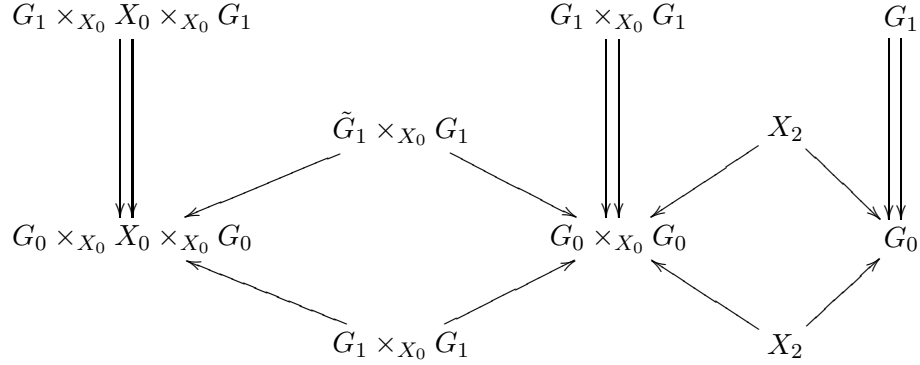
Identity Now we notice that $s_0 : X_0 \hookrightarrow G_0$ and $e_G \circ s_0 : X_0 \hookrightarrow G_1$, with e_G the identity of G , form a groupoid morphism from $X_0 \rightrightarrows X_0$ to $G_1 \rightrightarrows G_0$. This gives a morphism $\bar{e} : X_0 \rightarrow \mathcal{G}$ on the level of stacks.

Lemma 4.7. \bar{e} is the identity of \mathcal{G} .

Proof. Recall from Def. 3.4 that we need to show that there is a 2-morphism b_l between the two maps $m \circ (\bar{e} \times id)$ and $pr_2 : X_0 \times_{X_0, \bar{t}} \mathcal{G} \rightarrow \mathcal{G}$, and similarly a 2-morphism b_r . In our case, the H.S. bibundles presenting these two maps are $X_2|_{X_0 \times_{X_0, \bar{t}} G_0}$ and G_1 respectively and they are the same by construction, hence $b_l = id$. For b_r , by Remark 4.4, we have $X_2|_{G_0 \times_{s, X_0} X_0} = \tilde{G}_1$, so the isomorphism $\varphi^{-1} : \tilde{G}_1 \rightarrow G_1$ is b_r . Item 4b is implied by $s_0^1 \circ s_0^0 = s_1^1 \circ s_0^0$.

By Remark 3.6, we only need to show item 4e. Translating it into the language of

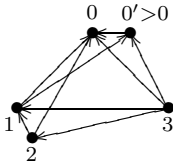
groupoids and bibundles, we obtain



Corresponding to item 4e, we need to show that the following diagrams commute:

$$\begin{array}{ccc}
 (\tilde{G}_1 \times_{X_0} G_1) \times_{G_0 \times_{X_0} G_0} X_2 / \sim & \xrightarrow{a} & (G_1 \times_{X_0} G_1) \times_{G_0 \times_{X_0} G_0} X_2 / \sim \\
 \downarrow b_r & \swarrow b_l & \\
 (G_1 \times_{X_0} G_1) \times_{G_0 \times_{X_0} G_0} X_2 / \sim & & \\
 \\
 [(\eta_3 = \eta_{012}, 1, \eta_1 = \eta_{023})] & \xrightarrow{\quad} & [(1, \eta_{123} = \eta_0, \eta_{013} = \eta_2)] \\
 \downarrow & \swarrow id? & \\
 [(\eta_{00'1}, 1, \eta_1)] & &
 \end{array}$$

Let us explain the diagram: An element $[(\eta_3, 1, \eta_1)] \in (\tilde{G}_1 \times_{X_0} G_1) \times_{G_0 \times_{X_0} G_0} X_2 / \sim$ fits inside picture (28)¹⁷, and its image under a is $[(1, \eta_{123}, \eta_{013})]$.



In this picture, $0' \rightarrow 0, 2 \rightarrow 1$ are degenerate edges and all the faces containing one of them are degenerate except for $\eta_{00'1}, \eta_{012}$ and η_{123} . (28)

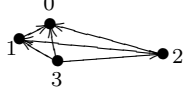
By the construction of b_r , $\eta_{00'1} = b_r(\eta_3)$. We only need to show that $\eta_{00'1} \cdot \eta_2 = \eta_0 \cdot \eta_1$. This is implied by the following: We consider the 3-simplices $(0, 0', 2, 3)$, $(0', 1, 2, 3)$ and $(0, 0', 1, 3)$, we have $\eta_1 = \eta_{0'23}$, $\eta_0 \cdot \eta_{0'23} = \eta_{0'13}$ and $\eta_{00'1} \cdot \eta_2 = \eta_{0'13}$ accordingly.

□

Inverse By Prop. 3.15, we only have to show that the actions of G and G^{op} on $X_2 \times_{d_1^2, G_0} M$, induced respectively by the first and second components of the left action of $G_1 \times_M G_1 \Rightarrow G_0 \times_M G_0$, are principal (see (13)). We prove this for the first copy of G_1 ;

¹⁷Now to avoid confusion, we call a face by its three vertices, for example now $\eta_1 = \eta_{023}$.

the proof for the second is similar. An element $(\eta_3, \eta_1, s_0 \circ s_0(d_1^1 \circ d_1^2(\eta_1))) \in G_1 \times_{s_G, G_0, d_2^2} X_2 \times_{d_1^2, G_0} M$ fits inside the following picture:



In this picture, $\eta_2 = s_0 \circ s_0(d_1^1 \circ d_1^2(\eta_1))$ is the degenerate face corresponding to the point 0 ($= 1 = 3$).

Then the freeness of the action is implied by $Kan!(3, 0)$, and the transitivity of the action is implied by $Kan(3, 0)$.

Comments on the étale condition If the $X_2 \rightarrow \Lambda[2, j](X)$ are étale maps between smooth manifolds, then the moment map $J_l : E_m \cong X_2 \rightarrow G_0 \times_{s, M, t} G_0$ is étale. By principality of the right action of the Lie groupoid $G_1 \rightrightarrows G_0$, it is an étale groupoid. This concludes the proof of the Theorem 4.2.

4.3 One-to-one correspondence

In this section, we use two lemmas to prove the following theorem:

Theorem 4.8. *There is a 1–1 correspondence between 2-groupoid objects in $(\mathcal{C}, \mathcal{T}'')$ modulo 1-Morita equivalence and those stacky groupoid objects in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$ whose identity maps have good charts.*

By Section 3.1, good charts (respectively good étale charts) always exist for SLie groupoids (respectively W-groupoids), so we have

Theorem 4.9. *There is a 1–1 correspondence between Lie 2-groupoids (respectively 2-étale Lie 2-groupoids) modulo 1-Morita equivalence and SLie groupoids (respectively W-groupoids).*

W-groupoids are isomorphic if and only if they are isomorphic as SLie groupoids, and 1-Morita equivalent 2-étale Lie 2-groupoids are 1-Morita equivalent Lie 2-groupoids. Therefore the étale version of the theorem is implied by the general case and we only have to prove the general case.

For the lemma below, we fix our notation: X and Y are 2-groupoid objects in $(\mathcal{C}, \mathcal{T}'')$ in the sense of Prop-Def. 2.16; $G_0 = X_1$ and $H_0 = Y_1$; $X_0 = Y_0 = M$. G_1 and H_1 are the spaces of bigons in X and Y , namely $d_2^{-1}(s_0(X_0))$ and $d_2^{-1}(s_0(Y_0))$ respectively; both $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ are groupoid objects, and they present presentable stacks \mathcal{G} and \mathcal{H} respectively. Moreover $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows M$ are stacky groupoids.

Lemma 4.10. *If $f : Y \rightarrow X$ is a hypercover, then*

1. *the groupoid $H_1 \rightrightarrows H_0$ constructed from Y satisfies $H_1 \cong G_1 \times_{G_0 \times_M G_0} H_0 \times H_0$ with the pull-back groupoid structure (therefore $\mathcal{G} \cong \mathcal{H}$);*
2. *the above map $\bar{\phi} : \mathcal{H} \cong \mathcal{G}$ induces a stacky groupoid isomorphism; that is,*
 - (a) *there are a 2-morphism $a : \bar{\phi} \circ m_{\mathcal{H}} \rightarrow m_{\mathcal{G}} \circ (\bar{\phi} \times \bar{\phi}) : \mathcal{H} \times_M \mathcal{H} \rightarrow \mathcal{G}$ and a 2-morphism $b : \bar{\phi} \circ \bar{e}_{\mathcal{H}} \rightarrow \bar{e}_{\mathcal{G}} : M \rightarrow \mathcal{G}$;*

(b) between maps $\mathcal{H} \times_M \mathcal{H} \times_M \mathcal{H} \rightarrow \mathcal{G}$, there is a commutative diagram of 2-morphisms

$$\begin{array}{ccc} \bar{\phi} \circ m_{\mathcal{H}} \circ (m_{\mathcal{H}} \times id) & \xrightarrow{a_H} & \bar{\phi} \circ m_{\mathcal{H}} \circ (id \times m_{\mathcal{H}}) \\ \downarrow a & & \downarrow a \\ m_{\mathcal{G}} \circ (m_{\mathcal{G}} \times id) \circ \bar{\phi}^{\times 3} & \xrightarrow{a_G} & m_{\mathcal{G}} \circ (id \times m_{\mathcal{G}}) \circ \bar{\phi}^{\times 3}, \end{array}$$

where by abuse of notation a denotes the 2-morphisms generated by a such as $a \odot (a \times id)$;

(c) between maps $M \times_M \mathcal{H} \rightarrow \mathcal{G}$ and maps $\mathcal{H} \times_M M \rightarrow \mathcal{G}$ there are commutative diagrams of 2-morphisms

$$\begin{array}{ccc} \bar{\phi} \circ m_{\mathcal{H}} \circ (\bar{e}_{\mathcal{H}} \times id) & \xrightarrow{b_l^H} & \bar{\phi} \circ pr_2 \\ \downarrow a \odot b & & \downarrow id \\ m_{\mathcal{G}} \circ (\bar{e}_{\mathcal{G}} \times id) \circ (id \times \bar{\phi}) & \xrightarrow{b_l^G} & pr_2 \circ (id \times \bar{\phi}) \end{array} \quad \begin{array}{ccc} m_{\mathcal{H}} \circ (id \times \bar{e}_{\mathcal{H}}) & \xrightarrow{b_r^H} & \bar{\phi} \circ pr_1 \\ \downarrow a \odot b & & \downarrow id \\ m_{\mathcal{G}} \circ (id \times \bar{e}_{\mathcal{G}}) \circ (\bar{\phi} \times id) & \xrightarrow{b_r^G} & pr_2 \circ (id \times \bar{\phi}). \end{array}$$

Proof. Since $Y_2 \cong \text{hom}(\partial\Delta^2, Y) \times_{\text{hom}(\partial\Delta^2, X)} X_2$, we have

$$\begin{aligned} H_1 &= d_2^{-1}(s_0(Y_0)) = d_2^{-1}(s_0(X_0)) \times_{d_1 \times d_0, X_1 \times_M X_1} Y_1 \times_M Y_1 \\ &= G_1 \times_{\mathbf{t}_G \times \mathbf{s}_G, G_0 \times_M G_0} H_0 \times_M H_0 \\ &= G_1 \times_{\mathbf{t}_G \times \mathbf{s}_G, G_0 \times G_0} H_0 \times H_0, \end{aligned}$$

where the last step follows from the facts that $(\mathbf{t}_G \times \mathbf{s}_G)(G_1) \subset G_0 \times_M G_0$ and that f preserves simplicial structures. The multiplication on H_1 (respectively G_1) is given by 3-multiplications on Y_2 (respectively X_2). Therefore H has the pull-back groupoid structure since Y is the pull-back of X . So item 1 is proven. We denote by $\phi_i : H_i \rightarrow G_i$ the groupoid morphism. Here $\phi_0 = f_1$, and ϕ_1 is a restriction of f_2 .

To prove 2a, we translate it into the following groupoid diagram:

$$\begin{array}{ccccc} & H_1 \times_M H_1 & & E_{m_{\mathcal{H}}} = Y_2 & \\ & \downarrow \Downarrow & & \searrow & \downarrow \Downarrow \\ G_1 \times_M G_1 & H_0 \times_M H_0 & \xleftarrow{E_{m_{\mathcal{G}}} = X_2} & & G_1 & \rightarrow H_0 \\ \downarrow \Downarrow & & \swarrow & & \downarrow \Downarrow & \swarrow \\ G_0 \times_M G_0 & & & & G_0 & \rightarrow H_0 \end{array}$$

We need to show that the following compositions of bibundles are isomorphic:

$$\begin{aligned} (Y_2 \times_{G_0} G_1)/H_1 &= (Y_2 \times_{H_0} H_0 \times_{G_0} G_1)/H_1 \\ &\stackrel{a}{\cong} H_0 \times_M H_0 \times_{G_0 \times_M G_0} X_2 = (H_0 \times_M H_0 \times_{G_0 \times_M G_0} G_1 \times_M G_1 \times_{G_0 \times_M G_0} X_2)/G_1 \times_M G_1. \end{aligned} \tag{29}$$

By item 1, any element in $(Y_2 \times_{G_0} G_1)/H_1$ can be written as $[(\eta, 1)]$ with $\eta \in Y_2$, and we construct a by $[(\eta, 1)] \mapsto (d_2(\eta), d_0(\eta), f_2(\eta))$. First of all, we need to show that a

is well-defined. For this we only have to notice that any element in Y_2 has the form $\eta = (\bar{\eta}, h_0, h_1, h_2)$ with $\bar{\eta} = f_2(\eta) \in X_2$ and $h_i = d_i(\eta)$, since f_2 preserves degeneracy maps. Also $\gamma \in H_1$ can be written as $\gamma = (\bar{\gamma}, h_1, h'_1)$ with $\bar{\gamma} = f_2(\gamma) \in G_1$; then the H_1 action on Y_2 is induced in the following way,

$$(\bar{\eta}, h_0, h_1, h_2) \cdot (\bar{\gamma}, h_1, h'_1) = (\bar{\eta} \cdot \bar{\gamma}, h_0, h'_1, h_2)$$

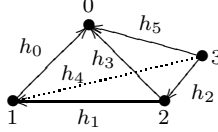
where $h_i = d_i(\eta)$. Hence if $(\eta', 1) = (\eta, 1) \cdot (\bar{\gamma}, h_1, h'_1)$, then $\bar{\gamma} = 1$ and $a([(\eta', 1)]) = (h_2, h_0, \bar{\eta}) = a([(\eta, 1)])$. Given $(h_2, h_0, \bar{\eta}) \in H_0 \times_M H_0 \times_{G_0 \times_M G_0} X_2$, take any h_1 such that $f_1(h_1) = d_1(\bar{\eta})$; then $(h_0, h_1, h_2) \in \text{hom}(\partial \Delta^2, Y)$. Thus we construct a^{-1} by $(h_2, h_0, \bar{\eta}) \mapsto [((\bar{\eta}, h_0, h_1, h_2), 1)]$. By the action of H_1 , it is easy to see that a^{-1} is also well-defined. For the 2-morphism b , the proof is much easier, since in this case all the H.S. morphisms are strict morphisms of groupoids. So we only have to use the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & G_0 \\ \downarrow & \nearrow & \\ H_0 & & \end{array}$$

Recall that the 3-multiplications on Y_2 are induced from those of X_2 in the following way:

$$m_0^Y((\bar{\eta}_1, h_2, h_5, h_3), (\bar{\eta}_2, h_4, h_5, h_0), (\bar{\eta}_3, h_1, h_3, h_0)) = (m_0^X(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3), h_2, h_4, h_1),$$

and similarly for other m^Y 's.



Then we have a diagram of 2-morphisms between composed bibundles

$$\begin{array}{ccc} ((Y_2 \times_M H_1 \times_{H_0^2} Y_2/H_1^2) \times_{G_0} G_1)/H_1 & \xrightarrow{a_H} & ((H_1 \times_M Y_2 \times_{G_0^2} Y_2)/H_1^2) \times_{G_0} G_1/H_1 \\ \downarrow a & & \downarrow a \\ (H_0^3 \times_{G_0^3} (X_2 \times_M G_1 \times_{G_0^2} X_2))/G_1^2 & \xrightarrow{a_G} & ((H_0^3 \times_{G_0^3} (G_1 \times_M X_2 \times_{G_0^2} X_2))/G_1^2, \end{array}$$

which is

$$\begin{array}{ccc} [(\eta_3, 1, \eta_2, 1)] & \xrightarrow{a_H} & [(1, \eta_0, \eta_2, 1)] \\ \downarrow a & & \downarrow a \\ [(h_0, h_1, h_2, \bar{\eta}_3, 1, \bar{\eta}_1)] & \xrightarrow{a_G?} & [(h_0, h_1, h_2, 1, \bar{\eta}_0, \bar{\eta}_2)] \end{array}$$

where $\bar{\eta}_i = f_2(\eta_i)$. Here \square^k denotes a suitable k -times fibre product of \square over M . Notice that f_2 preserves the 3-multiplications if and only if $m_0(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) = \bar{\eta}_0$. Since $a_G([(\bar{\eta}_3, 1, \bar{\eta}_1)]) = [(1, m_0(\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3), \bar{\eta}_2)]$, we conclude that f_2 preserves the 3-multiplications if and only if the above diagram commutes. So 2b is also proven.

Translating the right diagram in item 2c into groupoid language, we have

$$\begin{array}{ccc}
(J_l^{-1}(H_0 \times_M M) \times_{G_0} G_1)/H_1 & \xrightarrow{\text{restriction of } a} & H_0 \times_M M \times_{G_0 \times_M M} J_l^{-1}(G_0 \times_M M)/G_1 \times_M M \\
\downarrow b_r^H & & \downarrow b_r^G \\
(H_1 \times_{G_0} G_1)/H_1 & \xrightarrow{\phi_1} & H_0 \times_M M \times_{G_0 \times_M M} G_1/G_1 \times_M M
\end{array} \tag{30}$$

where J_l denotes the left moment map of X_2 or Y_2 to $G_0 \times_M G_0$ or $H_0 \times_M H_0$. The maps are explicitly: $[(\eta, 1)] \xrightarrow{a} [(h_2, s_0(x), f_2(\eta))] \xrightarrow{b_r^G} [(h_2, s_0(x), b_r^G(f_2(\eta)))]$ and $[(\eta, 1)] \xrightarrow{b_r^H} [(b_r^H(\eta), 1)] \xrightarrow{\phi_1} [(h_2, s_0(x), \phi_1(b_r^H(\eta)))]$, where $x = d_1(h_2)$. To show the commutativity of the diagram, we need to show that these two maps are the same; that is, we need to show $b_r^G(f_2(\eta)) = f_2(b_r^H(\eta))$, since ϕ_1 is a restriction of f_2 . Since $b_r = \varphi^{-1}$ is constructed by m 's as in Section 4.2, f_2 commutes with the b_r 's. We have a similar diagram for the left diagram of 2c, which is trivially commutative since $b_l^{H,G} = id$ by the construction in Section 4.2. So we proved item (2c). \square

To establish the inverse argument, we fix again the notation: $\mathcal{G} \Rightarrow M$ is a stacky groupoid object in $(\mathcal{C}, \mathcal{T}, \mathcal{T}')$; $G_1 \Rightarrow G_0$ and $H_1 \Rightarrow H_0$ are two groupoids in $(\mathcal{C}, \mathcal{T}'')$ presenting \mathcal{G} ; X and Y are the 2-groupoids corresponding to G and H as constructed in Section 4.1.

Lemma 4.11. *If there is a groupoid equivalence $\phi_i : H_i \rightarrow G_i$ in $(\mathcal{C}, \mathcal{T}'')$, then there is a 2-groupoid 1-hypercover $Y \rightarrow X$ in $(\mathcal{C}, \mathcal{T}'')$.*

Proof. Since both H and G present \mathcal{G} , which is a stacky groupoid over M , we are in the situation described in item 2 in Lemma 4.10; that is, we have a 2-morphism a satisfying various commutative diagrams as in items 2a, 2b, 2c. We take f_0 to be the isomorphism $M \cong M$, f_1 the map ϕ_0 , $f_2 : Y_2 \rightarrow X_2$ the map $\eta \mapsto [(\eta, 1)] \xrightarrow{a} (h_2, h_0, \bar{\eta}) \mapsto \bar{\eta}$ (see (29)). Since f_2 is made up of composition of morphisms, it is a morphism in \mathcal{C} . Since $d_2 \times d_0$ is the moment map and a preserves the moment map, we have $h_i = d_i(\eta)$ for $i = 0, 2$. It is clear that f_0 and f_1 preserve the structure maps since they preserve $\bar{e}, \bar{s}, \bar{t}$ of $\mathcal{G} \Rightarrow M$. It is also clear that $d_i f_2(\eta) = f_1(h_i)$ for $i = 2, 0$ since $(h_2, h_0, \bar{\eta} = f_2(\eta)) \in H_0^2 \times_{G_0^2} X_2$. Since a preserves moment maps, $f_1(d_1(\eta)) = J_r([(\eta, 1)]) = J_r(h_2, h_0, \bar{\eta}) = d_1(\bar{\eta})$, where J_r is the moment map to G_0 of the corresponding bibundles. Hence f_2 preserves the degeneracy maps.

For the face maps $s_0^1, s_1^1 : \square_1 \rightarrow \square_2$, we recall that $s_1^1(h) = (b_r^H)^{-1} e_H(h)$. Using the commutative diagram (30), by the definition of f_2 and the fact that $\phi_1 e_H = e_G \phi_0$, we have

$$f_2(s_1^1(h)) = pr_X a([(b_r^H)^{-1} e_H(h), 1]) = (b_r^G)^{-1} \phi_1 e_H(h) = (b_r^G)^{-1} e_G \phi_0(h) = s_1^1 f_1(h),$$

where pr_X is the natural map $H_0^2 \times_{G_0^2} X_2 \rightarrow X_2$. We treat s_0^1 similarly using the diagram for b_l . Hence f_2 preserves the face maps.

The fact that f_2 preserves the 3-multiplications follows from the proof of item (2b) of Lemma 4.10.

Then the induced map $\phi : Y_2 \rightarrow \text{hom}(\partial\Delta^2, Y) \times_{\text{hom}(\partial\Delta^2, X)} X_2$ is $\eta \mapsto [(\eta, 1)] \xrightarrow{a} (h_2, h_0, \bar{\eta}) \mapsto (h_0, h_1, h_2, \bar{\eta})$, where $\bar{\eta} = f_2(\eta)$ and $h_i = d_i(\eta)$. As a composition of morphisms, ϕ is a morphism in \mathcal{C} . Moreover ϕ is injective since a is injective. For any $(h_0, h_1, h_2, \bar{\eta}) \in \text{hom}(\partial\Delta^2, Y) \times_{\text{hom}(\partial\Delta^2, X)} Y_2$, we have $(h_0, h_2, \bar{\eta}) \in H_0 \times_M H_0 \times_{G_0 \times_M G_0} X_2$. Then there is

an η such that $[(\eta, 1)] = a^{-1}(h_0, h_2, \bar{\eta})$. Thus $\phi(\eta) = (h_0, d_1(\eta), h_2, \bar{\eta})$, which implies that $f_1(d_1(\eta)) = d_1(\bar{\eta}) = f_1(h_1)$. Therefore $(1, d_1(\eta), h_1) \in H_1$ and $d_i(\eta \cdot (1, d_1(\eta), h_1)) = h_i$, $i = 0, 1, 2$, since d_1 is the moment map to H_0 of the bibundle Y_2 . So $\phi(\eta \cdot (1, d_1(\eta), h_1)) = (\bar{\eta}, h_0, h_1, h_2)$, which shows the surjectivity. Therefore ϕ is an isomorphism. \square

The theorem is now proven, since we only have to consider the case when (1-) Morita equivalence is given by strict (2-) groupoid morphisms.

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